

# Davie's type uniqueness for a class of SDEs with jumps

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**Abstract:** A result of A.M. Davie [Int. Math. Res. Not. 2007] states that a multidimensional stochastic equation  $dX_t = b(t, X_t) dt + dW_t$ ,  $X_0 = x$ , driven by a Wiener process  $W = (W_t)$  with a coefficient  $b$  which is only bounded and measurable has a unique solution for almost all choices of the driving Wiener path. We consider a similar problem when  $W$  is replaced by a Lévy process  $L = (L_t)$  and  $b$  is  $\beta$ -Hölder continuous in the space variable,  $\beta \in (0, 1)$ . We assume that  $L_1$  has a finite moment of order  $\theta$ , for some  $\theta > 0$ . Using also a new càdlàg regularity result for strong solutions, we prove that strong existence and uniqueness for the SDE together with  $L^p$ -Lipschitz continuity of the strong solution with respect to  $x$  imply a Davie's type uniqueness result for almost all choices of the Lévy path. We apply this result to a class of SDEs driven by non-degenerate  $\alpha$ -stable Lévy processes,  $\alpha \in (0, 2)$  and  $\beta > 1 - \alpha/2$ .

**Keywords:** stochastic differential equations - Lévy processes - path-by-path uniqueness - Hölder continuous drift.

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## 1 Introduction

In [8] A.M. Davie has proved that a SDE  $dX_t = b(t, X_t) dt + dW_t$ ,  $X_0 = x \in \mathbb{R}^d$ , driven by a Wiener process  $W$  and having a coefficient  $b$  which is only bounded and measurable has a unique solution for almost all choices of the driving Wiener path. This type of uniqueness is also called *path-by-path uniqueness*. In other words, adding a single path of a Wiener process  $W = (W_t) = (W_t)_{t \geq 0}$  regularizes a singular ODE whose right-hand side  $b$  is only bounded and measurable.

We consider a similar uniqueness problem for SDEs driven by Lévy noises with Hölder continuous drift term  $b$ , i.e., we deal with

$$X_t(\omega) = x + \int_s^t b(r, X_r(\omega)) dr + L_t(\omega) - L_s(\omega), \quad t \in [s, T], \quad (1.1)$$

where  $T > 0$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $d \geq 1$ ,  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is measurable, bounded and  $\beta$ -Hölder continuous in the  $x$ -variable, uniformly in  $t$ ,  $\beta \in (0, 1]$ . Moreover

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$L = (L_t)$  is a  $d$ -dimensional Lévy process defined on a probability space  $(\Omega, \mathcal{F}, P)$  and  $\omega \in \Omega$  (see Section 2; recall that  $L_0 = 0$ ,  $P$ -a.s). Suppose that  $E[|L_1|^\theta] < \infty$  for some  $\theta > 0$  (cf. Hypothesis 2). Assuming that, for any  $x \in \mathbb{R}^d$ ,  $s \in [0, T]$ , strong existence and uniqueness hold for (1.1) together with  $L^p$ -Lipschitz continuity of the strong solution  $(X_t^{s,x})$  with respect to  $x$ , i.e.,

$$\sup_{s \in [0, T]} E \left[ \sup_{s \leq r \leq T} |X_r^{s,x} - X_r^{s,y}|^p \right] \leq C |x - y|^p, \quad x, y \in \mathbb{R}^d, \quad p \in [2, \infty) \quad (1.2)$$

(cf. Hypothesis 1 and Section 2) we prove the following result (cf. Theorem 5.1)

**Theorem 1.1.** *Assume Hypotheses 1 and 2. There exists an event  $\Omega' \in \mathcal{F}$  with  $P(\Omega') = 1$  such that for any  $\omega \in \Omega'$ ,  $x \in \mathbb{R}^d$ , the integral equation*

$$f(t) = x + \int_0^t b(r, f(r) + L_r(\omega)) dr, \quad t \in [0, T], \quad (1.3)$$

*has exactly one solution  $f$  in  $C([0, T]; \mathbb{R}^d)$ .*

The assumptions and the uniqueness property are clear when  $\beta = 1$  (the Lipschitz case). When  $\beta \in (0, 1)$  the result is a special case of assertion (v) in Theorem 5.1 which also considers  $s \neq 0$ . It turns out that  $f(t) = \phi(0, t, x, \omega) - L_t(\omega)$ ,  $t \in [0, T]$ , where  $(\phi(s, t, x, \cdot))$  is a particular strong solution to (1.1). In Section 6 we will apply the previous theorem to a class of SDEs driven by non-degenerate  $\alpha$ -stable type Lévy processes,  $\alpha \in (0, 2)$ , assuming as in [24] that  $\beta \in (1 - \frac{\alpha}{2}, 1)$ . Note that we can also treat locally Hölder drifts  $b(x)$  by a localization procedure (see Corollaries 5.4 and 5.5). These uniqueness results seem to be new even in dimension one. For instance, one can consider

$$dX_t = \sqrt{|X_t|} dt + dL_t^{(\alpha)}, \quad X_0 = x \in \mathbb{R},$$

with a symmetric  $\alpha$ -stable process  $L^{(\alpha)} = (L_t^{(\alpha)})$ ,  $\alpha > 1$ , and prove that for almost all  $\omega \in \Omega$  there exists at most one solution for (1.3) with  $b(r, x) = \sqrt{|x|}$  and  $L = L^{(\alpha)}$ .

As already mentioned when  $L = W$  is a standard Wiener process, Theorem 1.1 is a special case of Theorem 1.1 in [8]. Recall that Davie's uniqueness is stronger than the usual pathwise uniqueness considered in the literature on SDEs (cf. Remark 2.2 and see also [10]). Pathwise uniqueness deals with solutions which are adapted stochastic processes and does not consider solutions corresponding to single paths  $(L_t(\omega))_{t \in [0, T]}$ . When  $L = W$  several results on strong existence and pathwise uniqueness are known for the SDE (1.1) with very irregular drift  $b$ : the seminal paper [35] deals with  $b$  as in the Davie's result; further recent results consider  $b$  which is only locally in some  $L^p$ -spaces (see also [13], [18] and [9]).

When  $L$  is a stable type Lévy process, the SDE (1.1) with a Hölder continuous and bounded drift  $b$  and its associated integro-differential generator  $\mathcal{L}_b$  (cf. (6.8)) has received a lot of attention (see, for instance, [34], [24], [31], [32], [3], [25], [6] and the references therein). On this respect in Theorem 3.2 of [34] the authors proved that when  $d = 1$  and  $L$  is a symmetric  $\alpha$ -stable process,  $\alpha \in (0, 1)$ , pathwise uniqueness may fail even with a  $\beta$ -Hölder continuous  $b$  if  $\alpha + \beta < 1$ .

Let us come back to Davie's theorem. The proof in [8] is self-contained but very technical; it relies on explicit computations with Gaussian kernels. An alternative approach to the Davie uniqueness result has been proposed in [30] (see in particular

Theorems 1.1 and 3.1 in [30]). This approach uses the flow property of strong solutions of SDEs driven by the Wiener process. Beside [8] our work has been inspired by Theorem 3.1 in [30] which deals with drifts  $b$  possibly unbounded in time and such that  $b(t, \cdot)$  is Hölder continuous. We mention that applications of Davie's uniqueness to Euler approximations for (1.1) are given in Section 4 of [8].

In our proof we use  $L^p$ -estimates (1.2) which are well-known when  $L = W$  (they can be easily deduced from Section 2 in [12]). They are even true for more general drifts  $b$  (i.e.,  $b \in L^q(0, T; L^p(\mathbb{R}^d; \mathbb{R}^d))$ ,  $d/p + 2/q < 1$ ,  $p \geq 2$ ,  $q > 2$ , see formula (5.9) and Proposition 5.2 in [9]). Moreover, when  $L$  is a symmetric non-degenerate  $\alpha$ -stable process,  $b(t, x) = b(x)$ ,  $\alpha \geq 1$  and  $\beta \in (1 - \frac{\alpha}{2}, 1]$ , such estimates follow by Theorem 4.3 in [24] (see Theorem 6.6 for a more general case).

By the  $L^p$ -estimates (1.2), passing through different modifications (see Sections 3 and 4), we finally obtain a suitable strong solution  $\phi(s, t, x, \omega)$  (see Theorem 5.1) which solves (1.1) for any  $\omega \in \Omega'$ , for some almost sure event  $\Omega'$  which is independent on  $s, t$  and  $x$ . Such solution  $\phi$  is used to prove uniqueness of (1.3) (see the proof of (v) of Theorem 5.1). We also establish càdlàg regularity of  $\phi$  with respect to  $s$ , uniformly in  $t \in [0, T]$  and  $x$ , when  $x$  varies in compact sets of  $\mathbb{R}^d$ . This result seems to be new even when  $d = 1$  and  $b$  is Lipschitz continuous if  $L$  is not the Wiener process  $W$  (when  $L = W$ , the continuous dependence on  $s$ , uniformly in  $x$ , has been proved in Section 2 of [14] for SDEs with Lipschitz coefficients). We also prove the continuous dependence of  $\phi(s, t, x, \omega)$  with respect to  $x$  and the flow property, for any  $\omega \in \Omega'$  (see assertions (iii) and (iv) in Theorem 5.1). There are recent papers on the flow property for solutions to SDEs with jumps (see, for instance, [25], [21], [6] and the references therein). However they do not prove the previous assertions on  $\phi$ .

Remark that when  $L = W$  and  $b(t, \cdot)$  is Hölder continuous as in (1.1), proving the existence of a regular strong solution like  $\phi$  is easier. Indeed in such case one can use the well-known Kolmogorov-Chentsov continuity test to get a continuous dependence on  $(s, t, x)$ . More precisely, when  $L = W$ , we can apply the Zvonkin method of [35] or the related Itô-Tanaka trick of [12] and, using a suitable regular solution  $u(t, x)$  of a related Kolmogorov equation (cf. Section 6.2), find that the process  $(u(t, X_t^x))$  solves an auxiliary SDE with Lipschitz continuous coefficients. On this auxiliary equation one can perform the Kolmogorov-Chentsov test as in [19] and finally obtain the required regular modification of the strong solution. To get our regular strong solution  $\phi$  we do not pass through an auxiliary SDE but work directly on (1.1) using first a result in [14] and then a càdlàg criterion given in [4]. We apply this criterion to a suitable stochastic process with values in a space of continuous functions defined on  $\mathbb{R}^d$  (see Theorem 4.4). This approach could be also useful to study regularity properties of solutions to SDEs with multiplicative noise.

In Section 6 we apply Theorem 5.1 to a class of SDEs driven by non-degenerate  $\alpha$ -stable type Lévy processes, using also results in [24] and [25]. In particular we prove a Davie's type uniqueness result for (1.1) when  $L$  is a standard rotationally invariant  $\alpha$ -stable process,  $\alpha \in (0, 2)$  and  $\beta \in (1 - \frac{\alpha}{2}, 1]$ . The generator of  $L$  is the well-known fractional Laplacian  $-(-\Delta)^{\alpha/2}$ . To cover the case  $\alpha \in (0, 1)$  we also need an analytic result proved in [31] (cf. Remark 5.5 in [25]). When  $\alpha \in [1, 2)$  and  $\beta \in (1 - \frac{\alpha}{2}, 1]$  we can treat more general non-degenerate  $\alpha$ -stable type processes like relativistic and truncated stable processes and some tempered stable processes (cf. [25] with the references therein and see Examples 6.2). When  $\alpha \in [1, 2)$  we can also consider the singular  $\alpha$ -stable process  $L = (L_t)$ ,  $L_t = (L_t^1, \dots, L_t^d)$ ,  $t \geq 0$ , where  $L^1, \dots, L^d$  are

independent one-dimensional symmetric  $\alpha$ -stable processes; well-posedness of SDEs driven by this process has recently received particular attention (see, for instance, [2], [24], [38], [25], [6]).

## 2 Notations and assumptions

We fix basic notations. We refer to [28], [20], [17] and [1] for more details on Lévy processes with values in  $\mathbb{R}^d$ . By  $\langle x, y \rangle$  (or  $x \cdot y$ ) we denote the euclidean inner product between  $x$  and  $y \in \mathbb{R}^d$ , for  $d \geq 1$ ; further  $|x| = (\langle x, x \rangle)^{1/2}$ . If  $H \subset \mathbb{R}^d$  we denote by  $1_H$  its indicator function. The Borel  $\sigma$ -algebra of a Borel set  $C \subset \mathbb{R}^k$ ,  $k \geq 1$ , is indicated by  $\mathcal{B}(C)$ . Similarly if  $(S, d)$  is a metric space we denote its Borel  $\sigma$ -algebra by  $\mathcal{B}(S)$ . We consider a complete probability space  $(\Omega, \mathcal{F}, P)$ . The expectation with respect to  $P$  is indicated with  $E$ . If  $\mathcal{G} \subset \mathcal{F}$  is a  $\sigma$ -algebra, a random variable  $X : \Omega \rightarrow S$  with values in a metric space  $(S, d)$  which is measurable from  $(\Omega, \mathcal{G})$  into  $(S, \mathcal{B}(S))$  is called  $\mathcal{G}$ -measurable. Similarly a function  $l : [0, T] \times \Omega \rightarrow S$  is  $\mathcal{B}([0, T]) \times \mathcal{F}$ -measurable if  $l$  is measurable with respect to the product  $\sigma$ -algebra  $\mathcal{B}([0, T]) \times \mathcal{F}$ .

In the sequel we often need to specify the possible dependence of events of probability one from some parameters. Recall that a set  $\Omega' \subset \Omega$  is an *almost sure event* if  $\Omega' \in \mathcal{F}$  and  $P(\Omega') = 1$ . To stress that  $\Omega'$  possibly depends also on a parameter  $\lambda$  we write  $\Omega'_\lambda$  (the almost sure event  $\Omega'_\lambda$  may change from one proposition to another); for instance the notation  $\Omega_{s,x}$  means that the almost sure event  $\Omega_{s,x}$  possibly depends also on  $s$  and  $x$ . We say that a property involving random variables holds on an almost sure event  $\Omega'$  to indicate that such property holds for any  $\omega \in \Omega'$  (i.e., such property holds  $P$ -a.s.).

A  $d$ -dimensional stochastic process  $L = (L_t) = (L_t)_{t \geq 0}$ ,  $d \geq 1$ , defined on  $(\Omega, \mathcal{F}, P)$  is a *Lévy process* if it has independent and stationary increments, càdlàg paths (i.e.,  $P$ -a.s., each mapping  $t \mapsto L_t(\omega)$  is càdlàg from  $[0, \infty)$  into  $\mathbb{R}^d$ ; we denote by  $L_{s-}(\omega)$  the left-limit in  $s > 0$ ) and  $L_0 = 0$ ,  $P$ -a.s..

Similarly to Chapter II in [19] and Chapter V in [17] we define for  $0 \leq s < t < \infty$  the  $\sigma$ -algebra  $\mathcal{F}_{s,t}^L$  as the completion of the  $\sigma$ -algebra generated by the random variables  $L_r - L_s$ ,  $r \in [s, t]$ . We also set  $\mathcal{F}_{0,t}^L = \mathcal{F}_t^L$ . Since  $L$  has independent increments we have that  $L_v - L_u$  is independent of  $\mathcal{F}_u^L$  for  $0 \leq u < v$ . Note that  $(\Omega, \mathcal{F}, (\mathcal{F}_t^L)_{t \geq 0}, P)$  is an example of stochastic basis which satisfies the usual assumptions (see [1, page 72]). Given a Lévy process  $L$  there exists a unique function  $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$  such that

$$E[e^{i\langle h, L_t \rangle}] = e^{-t\psi(h)}, \quad h \in \mathbb{R}^d, \quad t \geq 0;$$

$\psi$  is called the *exponent* of  $L$ . The Lévy-Khintchine formula for  $\psi$  states that

$$\psi(h) = \frac{1}{2} \langle Qh, h \rangle - i \langle a, h \rangle - \int_{\mathbb{R}^d} (e^{i\langle h, y \rangle} - 1 - i \langle h, y \rangle 1_{\{|y| \leq 1\}}(y)) \nu(dy), \quad (2.1)$$

$h \in \mathbb{R}^d$ , where  $Q$  is a symmetric non-negative definite  $d \times d$ -matrix,  $a \in \mathbb{R}^d$  and  $\nu$  is a  $\sigma$ -finite (Borel) measure on  $\mathbb{R}^d$ , such that  $\int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(dy) < \infty$ ,  $\nu(\{0\}) = 0$  ( $1 \wedge |y|^2 = \min(1, |y|^2)$ );  $\nu$  is the *Lévy measure* (or intensity measure) of  $L$ . The triplet  $(Q, \nu, a)$  uniquely identifies the law of  $L$  (see Proposition 9.8 in [28] or Corollary 2.4.21 in [1]). It is called *generating triplet* (or characteristics) of the Lévy process  $L$ .

Given two stochastic processes  $X = (X_t)_{t \in [0, T]}$  and  $Y = (Y_t)_{t \in [0, T]}$  defined on  $(\Omega, \mathcal{F}, P)$  and with values in a metric space  $(S, d)$ , we say that  $X$  is a *modification*

or version of  $Y$  if for any  $t \in [0, T]$ ,  $X_t = Y_t$ ,  $P$ -a.s.; if in addition both  $X$  and  $Y$  have càdlàg paths then,  $P(X_t = Y_t, t \in [0, T]) = P(X_t = Y_t, \text{ for any } t \in [0, T]) = 1$ .

Let  $L = (L_t)$  be a  $d$ -dimensional Lévy process defined on a complete probability space  $(\Omega, \mathcal{F}, P)$ , let  $s \in [0, T]$  and  $x \in \mathbb{R}^d$  and consider the SDE

$$dX_t = b(t, X_t)dt + dL_t, \quad s \leq t \leq T, \quad X_s = x, \quad (2.2)$$

with  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  which is a locally bounded Borel function.

According to [19], [20] and [33] we say that an  $\mathbb{R}^d$ -valued stochastic process  $U^{s,x} = (U_t^{s,x})_{t \in [s, T]}$  defined on  $(\Omega, \mathcal{F}, P)$  is a *strong solution* to (2.2) starting from  $x$  at time  $s$  if, for any  $t \in [s, T]$ , the random variable  $U_t^{s,x} : \Omega \rightarrow \mathbb{R}^d$  is  $\mathcal{F}_{s,t}^L$ -measurable; further we require that there exists an almost sure event  $\Omega_{s,x}$  (possibly depending also on  $s$  and  $x$  but independent of  $t$ ) such that the following conditions hold for any  $\omega \in \Omega_{s,x}$ : (i) the map:  $t \mapsto U_t^{s,x}(\omega)$  is càdlàg on  $[s, T]$ ; (ii) we have

$$U_t^{s,x}(\omega) = x + \int_s^t b(r, U_r^{s,x}(\omega))dr + L_t(\omega) - L_s(\omega), \quad t \in [s, T]; \quad (2.3)$$

(iii) the path  $t \mapsto L_t(\omega)$  is càdlàg and  $L_0(\omega) = 0$ .

Given a strong solution  $U^{s,x}$  we set for any  $0 \leq t \leq s$ ,  $U_t^{s,x} = x$  on  $\Omega$ .

Let us recall some function spaces used in the paper. We consider  $C_b(\mathbb{R}^d; \mathbb{R}^k)$ , for integers  $k, d \geq 1$ , as the Banach space of all continuous and bounded functions  $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$  endowed with the supremum norm  $\|g\|_0 = \|g\|_{C_b} = \sup_{x \in \mathbb{R}^d} |g(x)|$ ,  $g \in C_b(\mathbb{R}^d; \mathbb{R}^k)$ . Moreover,  $C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^k)$ ,  $\beta \in (0, 1]$ , is the subspace of all  $\beta$ -Hölder continuous functions  $g$ , i.e.,  $g$  verifies

$$[g]_{C_b^{0,\beta}} = [g]_\beta := \sup_{x \neq x' \in \mathbb{R}^d} (|g(x) - g(x')| |x - x'|^{-\beta}) < \infty$$

(when  $\beta = 1$ ,  $g$  is Lipschitz continuous). If  $\beta = 0$  we set  $C_b^{0,0}(\mathbb{R}^d; \mathbb{R}^k) = C_b(\mathbb{R}^d; \mathbb{R}^k)$ . If  $\beta \in (0, 1)$  we also write  $C_b^\beta(\mathbb{R}^d; \mathbb{R}^k) = C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^k)$ ; note that  $C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^k)$  is a Banach space with the norm  $\|\cdot\|_{C_b^{0,\beta}} = \|\cdot\|_\beta = \|\cdot\|_0 + [\cdot]_\beta$ ,  $\beta \in (0, 1]$ . If  $\mathbb{R}^k = \mathbb{R}$ , we set  $C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^k) = C_b^{0,\beta}(\mathbb{R}^d)$  (a similar convention is also used for other function spaces). A function  $g \in C_b(\mathbb{R}^d; \mathbb{R}^k)$  belongs to  $C_b^1(\mathbb{R}^d; \mathbb{R}^k)$  if it is differentiable on  $\mathbb{R}^d$  and its Fréchet derivative  $Dg \in C_b(\mathbb{R}^d; \mathbb{R}^{dk})$ . If  $\beta \in (0, 1)$ , a function  $g \in C_b^1(\mathbb{R}^d; \mathbb{R}^k)$  belongs to  $C_b^{1+\beta}(\mathbb{R}^d; \mathbb{R}^k)$  if  $Dg \in C_b^\beta(\mathbb{R}^d; \mathbb{R}^{dk})$ . The space  $C_b^{1+\beta}(\mathbb{R}^d; \mathbb{R}^k)$  is a Banach space endowed with the norm  $\|g\|_{1+\beta} = \|g\|_{C_b^{1+\beta}} = \|g\|_0 + [Dg]_\beta$ ,  $g \in C_b^{1+\beta}(\mathbb{R}^d; \mathbb{R}^k)$ .  $C_b^\infty(\mathbb{R}^d; \mathbb{R}^k)$  is the space of all infinitely differentiable functions from  $\mathbb{R}^d$  into  $\mathbb{R}^k$  with all bounded derivatives. Finally  $g \in C_b^\infty(\mathbb{R}^d)$  belongs to  $C_0^\infty(\mathbb{R}^d)$  if  $g$  has compact support. Given a bounded open set  $B \subset \mathbb{R}^d$  we can define similar Banach spaces  $C^\beta(B)$  and  $C^{1+\beta}(B)$  with norms  $\|\cdot\|_{C^\beta(B)}$  and  $\|\cdot\|_{C^{1+\beta}(B)}$ ,  $\beta \in (0, 1)$ .

We usually require that the drift  $b$  belongs to  $L^\infty(0, T; C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d))$ ,  $\beta \in [0, 1]$ . This means that  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is Borel measurable and bounded,  $b(t, \cdot) \in C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d)$ ,  $t \in [0, T]$ , and  $[b]_{\beta,T} = \sup_{t \in [0, T]} [b(t, \cdot)]_{C_b^{0,\beta}} < \infty$ .

Set  $\|b\|_{\beta,T} = [b]_{\beta,T} + \|b\|_0$ ,  $\|b\|_0 = \sup_{t \in [0, T], x \in \mathbb{R}^d} |b(t, x)|$  if  $\beta \in (0, 1]$  and  $\|b\|_{0,T} = \|b\|_0$ ,  $\beta = 0$ . Note that  $(L^\infty(0, T; C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d)), \|\cdot\|_{\beta,T})$  is a Banach space. We will also use

$$G_0 = C([0, T]; \mathbb{R}^d) \quad (2.4)$$

to denote the separable Banach space consisting of all continuous functions  $f : [0, T] \rightarrow \mathbb{R}^d$ , endowed with the usual supremum norm  $\|\cdot\|_{G_0}$ .

Let us formulate our assumptions on (1.1) when  $b \in L^\infty(0, T; C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d))$ ,  $\beta \in [0, 1]$ . Note that, possibly changing  $b(t, x)$  with  $b(t, x) + a$ , to study the SDE (1.1) we may always assume that in the generating triplet  $(Q, \nu, a)$  we have

$$a = 0. \quad (2.5)$$

In (1.1) we deal with a Lévy process  $L$  defined on  $(\Omega, \mathcal{F}, P)$  and  $b \in L^\infty(0, T; C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d))$  which satisfy

*Hypothesis 1.* (i) For any  $s \in [0, T]$  and  $x \in \mathbb{R}^d$  on  $(\Omega, \mathcal{F}, P)$  there exists a strong solution  $(U_t^{s,x})_{t \in [0, T]}$  to (2.2).

(ii) Let  $s \in [0, T]$ . Given any two strong solutions  $(U_t^{s,x})_{t \in [0, T]}$  and  $(U_t^{s,y})_{t \in [0, T]}$  defined on  $(\Omega, \mathcal{F}, P)$  which both solve (2.2) with respect to  $L$  and  $b$  (starting from  $x$  and  $y \in \mathbb{R}^d$ , respectively, at time  $s$ ) we have, for any  $p \geq 2$ ,

$$\sup_{s \in [0, T]} E \left[ \sup_{s \leq t \leq T} |U_t^{s,x} - U_t^{s,y}|^p \right] \leq C(T) |x - y|^p, \quad x, y \in \mathbb{R}^d, \quad (2.6)$$

with  $C(T) = C((\nu, Q, 0), \|b\|_{\beta, T}, d, \beta, p, T) > 0$  independent of  $s, x$  and  $y$ . ■

The previous hypothesis holds clearly for any Lévy process  $L$  if  $\beta = 1$  (the Lipschitz case). Next we consider the Lévy measure  $\nu$  associated to the large jump parts of  $L$ .

*Hypothesis 2.* There exists  $\theta > 0$  such that  $\int_{\{|x|>1\}} |x|^\theta \nu(dx) < \infty$ . ■

*Remark 2.1.* By Theorems 25.3 and 25.18 in [28] the following three conditions are equivalent:

- (a)  $\int_{\{|x|>1\}} |x|^\theta \nu(dx) < \infty$  for some  $\theta > 0$ ;
- (b)  $E[|L_t|^\theta] < \infty$  for some  $t > 0$ ;
- (c)  $E[\sup_{s \in [0, t]} |L_s|^\theta] < \infty$  for any  $t > 0$ .

Note also that  $\int_{\{|x|>1\}} |x|^\theta \nu(dx) < \infty$  holds for some  $\theta > 0$  then  $\int_{\{|x|>1\}} |x|^{\theta'} \nu(dx) < \infty$  for any  $\theta' \in (0, \theta]$ .

*Remark 2.2.* We present here for the sake of completeness some general concepts about solutions of SDEs (cf. [31] for more details). We will not use these notions in the sequel. Let the initial time  $s = 0$ . A weak solution to (1.1) with initial condition  $x \in \mathbb{R}^d$  is a tuple  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P, L, X)$ , where  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  is a stochastic basis on which it is defined a Lévy process  $L$  and a càdlàg  $(\mathcal{F}_t)$ -adapted  $\mathbb{R}^d$ -valued process  $X = (X_t)$  which solves (1.1)  $P$ -a.s.. A weak solution  $X$  which is  $(\mathcal{F}_t^L)$ -adapted is called strong solution. One say that pathwise uniqueness holds for (1.1) if given two weak solutions  $X$  and  $Y$  (starting from  $x \in \mathbb{R}^d$ ) and defined on the same stochastic basis (with respect to the same  $L$ ) then  $P$ -a.s. we have  $X_t = Y_t$ , for any  $t \in [0, T]$ .

### 3 Preliminary results on strong solutions

Consider (2.2) with  $b \in L^\infty(0, T; C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d))$ ,  $\beta \in [0, 1]$ , and suppose that  $L$  defined on  $(\Omega, \mathcal{F}, P)$  and  $b$  satisfy Hypothesis 1.

Let  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$ . We start with a strong solution  $(\tilde{X}_t^{s,x})_{t \in [0, T]}$  to (2.2) defined on  $(\Omega, \mathcal{F}, P)$  and introduce the  $d$ -dimensional process  $\tilde{Y}^{s,x} = (\tilde{Y}_t^{s,x})_{t \in [0, T]}$ ,

$$\tilde{Y}_t^{s,x} = \tilde{X}_t^{s,x} - (L_t - L_s), \quad t \geq s. \quad (3.1)$$

Note that on some almost sure event  $\Omega_{s,x}$  (independent of  $t$ ) we have

$$\tilde{Y}_t^{s,x} = x + \int_s^t b(r, \tilde{Y}_r^{s,x} + (L_r - L_s)) dr, \quad t \geq s, \quad (3.2)$$

and  $\tilde{Y}_t^{s,x} = x$  on  $\Omega$  if  $t \leq s$ . It follows that  $(\tilde{Y}_t^{s,x})_{t \in [0, T]}$  have *continuous paths*.

Let us fix  $s \in [0, T]$  and  $x \in \mathbb{R}^d$ . We modify the process  $\tilde{Y}^{s,x}$  only on  $\Omega \setminus \Omega_{s,x}$  by setting  $\tilde{Y}_t^{s,x}(\omega) = x$ , for  $t \in [0, T]$ , if  $\omega \notin \Omega_{s,x}$  (we still denote by  $\tilde{Y}^{s,x}$  such new process).

We find that  $\tilde{Y}^{s,x}(\omega) \in G_0 = C([0, T]; \mathbb{R}^d)$ , for any  $\omega \in \Omega$ . Moreover (cf. (2.4)) it is easy to check that

$$\tilde{Y}^{s,x} = \tilde{Y}^{s,x} \quad \text{is a random variable with values in } G_0. \quad (3.3)$$

Now, for each fixed  $s \in [0, T]$ , we will construct a suitable modification of the random field  $(\tilde{Y}^{s,x})_{x \in \mathbb{R}^d}$  with values in  $G_0$ . We need the following special case of Theorem 1.1 of [14]. It is a generalized Garsia-Rodemich-Rumsey type lemma.

**Theorem 3.1.** ([14]) *Let  $(M, \rho)$  be a separable metric space and  $(\Omega, \mathcal{F}, P)$  be a probability space. Let  $\psi : \Omega \times \mathbb{R}^d \rightarrow M$  be a  $\mathcal{F} \times \mathcal{B}(\mathbb{R}^d)$ -measurable map such that  $\psi(\omega, \cdot)$  is continuous on  $\mathbb{R}^d$ , for each  $\omega \in \Omega$ , and there exists  $c > 0$  and  $p > 2d$  for which  $E[(\rho(\psi(\cdot, x), \psi(\cdot, y)))^p] \leq c|x - y|^p$ ,  $x, y \in \mathbb{R}^d$ . Then, for any  $\omega \in \Omega$ ,  $x, y \in \mathbb{R}^d$ ,*

$$\rho(\psi(\omega, x), \psi(\omega, y)) \leq Y(\omega)|x - y|^{1 - \frac{2d}{p}} [(|x| \vee |y|)^{\frac{2d+1}{p}} \vee 1], \quad (3.4)$$

where  $Y : \Omega \rightarrow [0, \infty]$  is the following  $p$ -integrable random variable:

$$Y(\omega) = \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\rho(\psi(\omega, x), \psi(\omega, y)))^p}{|x - y|^p} f(x)f(y) dx dy \right)^{1/p}, \quad \omega \in \Omega,$$

with  $f(x) = c(d, p)([|x|^d ((\log(|x|) \vee 0)^2) \vee 1]^{-1})$ ,  $x \neq 0$ , for some constant  $c(d, p) > 0$ .

In Theorem 1.1 of [14]  $f(x)$  is just defined as  $([|x|^d ((\log(|x|) \vee 0)^2) \vee 1]^{-1})$ . Moreover  $Y(\omega) = c_3(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\rho(\psi(\omega, x), \psi(\omega, y)))^p}{|x - y|^p} f(x)f(y) dx dy)^{1/p}$ .

**Lemma 3.2.** *Consider (2.2) with  $b \in L^\infty(0, T; C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d))$ ,  $\beta \in [0, 1]$ , and suppose that  $L$  defined on  $(\Omega, \mathcal{F}, P)$  and  $b$  satisfy Hypothesis 1. Let us fix  $s \in [0, T]$  and consider the random field  $\tilde{Y}^s = (\tilde{Y}^{s,x})_{x \in \mathbb{R}^d}$  with values in  $G_0$  (see (3.3)). We have:*

(i) *There exists a continuous version  $Y^s = (Y^{s,x})_{x \in \mathbb{R}^d}$  with values in  $G_0$  (i.e., for any  $x \in \mathbb{R}^d$ ,  $Y^{s,x} = \tilde{Y}^{s,x}$  in  $G_0$  on some almost sure event).*

(ii) For any  $p > 2d$  there exists a random variable  $U_{s,p}$  with values in  $[0, \infty]$  such that, for any  $\omega \in \Omega$ ,  $x, y \in \mathbb{R}^d$ ,

$$\|Y^{s,x}(\omega) - Y^{s,y}(\omega)\|_{G_0} \leq U_{s,p}(\omega) [(|x| \vee |y|)^{\frac{2d+1}{p}} \vee 1] |x - y|^{1-2d/p}. \quad (3.5)$$

Moreover, with the same constant  $C(T)$  appearing in (2.6),

$$\sup_{s \in [0, T]} E[U_{s,p}^p] \leq C(d) C(T) < \infty \quad (3.6)$$

where  $C(d) = \left( \int_{\mathbb{R}^d} f(x) dx \right)^2$  (hence  $U_{s,p}$  is finite on some almost sure event possibly depending on  $s$  and  $p$ ).

(iii) On some almost sure event  $\Omega'_s$  (independent of  $t$  and  $x$ ) we have

$$Y_t^{s,x} = x + \int_s^t b(r, Y_r^{s,x} + (L_r - L_s)) dr, \quad t \geq s, \quad x \in \mathbb{R}^d \quad (3.7)$$

(where  $Y_t^{s,x}(\omega) = (Y^{s,x}(\omega))(t)$ ,  $t \in [0, T]$ ).

*Proof.* (i) Using (2.6) we can apply the Kolmogorov-Chentsov continuity test as in [15], page 57, and obtain a continuous version  $Y^s$  of  $\tilde{Y}^s$ . The classical proof given in [15] uses the Borel-Cantelli lemma; by such proof it is easy to show that an analogous of (2.6) holds for  $Y^s$ , i.e., for  $p \geq 2$ ,  $x, y \in \mathbb{R}^d$ ,

$$\sup_{s \in [0, T]} E[\|Y^{s,x} - Y^{s,y}\|_{G_0}^p] = \sup_{s \in [0, T]} E[\|\tilde{Y}^{s,x} - \tilde{Y}^{s,y}\|_{G_0}^p] \leq C(T) |x - y|^p. \quad (3.8)$$

(ii) As in Theorem 3.1 we consider the random variables

$$U_{s,p}(\omega) = \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{\|Y^{s,x}(\omega) - Y^{s,y}(\omega)\|_{G_0}}{|x - y|} \right)^p f(x) f(y) dx dy \right)^{1/p},$$

$\omega \in \Omega$ ,  $p > 2d$  and  $s \in [0, T]$ . By (3.8) and Theorem 3.1 we obtain (3.5) and (3.6).

(iii) We start from equation (3.2) involving the process  $(\tilde{Y}^{s,x})$ . Since for some almost sure event  $\Omega'_{s,x} \subset \Omega_{s,x}$ , we have  $Y_t^{s,x}(\omega) = \tilde{Y}_t^{s,x}(\omega)$ ,  $\omega \in \Omega'_{s,x}$ ,  $t \in [0, T]$ , we obtain from (3.2)

$$Y_t^{s,x}(\omega) = x + \int_s^t b(r, Y_r^{s,x}(\omega) + (L_r(\omega) - L_s(\omega))) dr,$$

for any  $s \in [t, T]$ ,  $x \in \mathbb{Q}^d$ ,  $\omega \in \Omega'_s = \bigcap_{x \in \mathbb{Q}^d} \Omega'_{s,x}$ . Note also that by (i) the function:  $x \mapsto Y^{s,x}(\omega)$  is continuous for all  $\omega \in \Omega$ . Take now  $x \in \mathbb{R}^d$  and let  $(x_n) \subset \mathbb{Q}^d$  be a sequence converging to  $x$ . It follows from the continuity of  $b(r, \cdot)$  and the dominated convergence theorem that, for any  $t \geq s$ , on  $\Omega'_s$  we have:

$$\begin{aligned} Y_t^{s,x} &= \lim_{n \rightarrow \infty} Y_t^{s,x_n} = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} \int_s^t b(r, Y_r^{s,x_n} + (L_r - L_s)) dr \\ &= x + \int_s^t b(r, Y_r^{s,x} + (L_r - L_s)) dr \end{aligned}$$

and this shows the assertion. ■



Let  $s \in [0, T]$ . According to the previous result starting from  $Y^s = (Y^{s,x})_{x \in \mathbb{R}^d}$  we can define random variables  $X_t^{s,x} : \Omega \rightarrow \mathbb{R}^d$  as follows:  $X_t^{s,x} = x$  if  $t \leq s$  and

$$X_t^{s,x} = Y_t^{s,x} + (L_t - L_s), \quad s, t \in [0, T], \quad x \in \mathbb{R}^d, \quad s \leq t. \quad (3.9)$$

By the properties of  $Y^{s,x}$  we get  $P(\tilde{X}_t^{s,x} = X_t^{s,x}, t \in [0, T]) = 1$ , for any  $x \in \mathbb{R}^d$  (cf. (3.1)). Moreover, using also (3.7), we find that for some almost sure event  $\Omega'_s$  (independent of  $x$  and  $t$ ) the map:  $t \mapsto X_t^{s,x}(\omega)$  is càdlàg on  $[0, T]$ , for any  $\omega \in \Omega'_s$ ,  $x \in \mathbb{R}^d$ , and on  $\Omega'_s$  we have

$$X_t^{s,x} = x + \int_s^t b(r, X_r^{s,x}) dr + L_t - L_s, \quad s \leq t \leq T, \quad x \in \mathbb{R}^d. \quad (3.10)$$

Thus  $(X_t^{s,x})_{t \in [0, T]}$  is a particular *strong solution* to (2.2). By Lemma 3.2 we also have, for any  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$ , on  $\Omega$

$$\lim_{y \rightarrow x} \sup_{t \in [0, T]} |X_t^{s,x} - X_t^{s,y}| = 0. \quad (3.11)$$

We can prove the following flow property.

**Lemma 3.3.** *Under the same assumptions of Lemma 3.2 consider the strong solution  $(X_t^{s,x})_{t \in [0, T]}$  defined in (3.9). Let  $0 \leq s < u \leq T$ . There exists an almost sure event  $\Omega_{s,u}$  (independent of  $t \in [u, T]$  and  $x \in \mathbb{R}^d$ ) such that for  $\omega \in \Omega_{s,u}$ ,  $x \in \mathbb{R}^d$ , we have*

$$X_t^{s,x}(\omega) = X_t^{u, X_u^{s,x}(\omega)}(\omega), \quad t \in [u, T], \quad x \in \mathbb{R}^d. \quad (3.12)$$

*Proof.* Let us fix  $s, u \in [0, T]$ ,  $s < u$ , and  $x \in \mathbb{R}^d$ . We introduce the process  $(V_t^x)_{0 \leq t \leq T}$  on  $(\Omega, \mathcal{F}, P)$  with values in  $\mathbb{R}^d$ :

$$V_t^x(\omega) = \begin{cases} X_t^{s,x}(\omega) & \text{for } 0 \leq t \leq u, \\ X_t^{u, X_u^{s,x}(\omega)}(\omega) & \text{for } u < t \leq T \end{cases}, \quad \omega \in \Omega.$$

In order to prove (3.12) we will show that  $(V_t^x)$  is strong solution to (2.2) for  $t \geq s$ . Then by uniqueness we will get the assertion.

It is easy to prove that  $(V_t^x)$  has càdlàg paths. More precisely, by (3.7) on some almost sure event  $\Omega'_s \cap \Omega'_u$  (independent of  $x$ ) we have that  $t \mapsto V_t^x(\omega)$  is càdlàg on  $[0, T]$  (note also that, for any  $\omega \in \Omega'_s \cap \Omega'_u$ ,  $z \in \mathbb{R}^d$ ,  $\lim_{t \rightarrow u^+} X_t^{u,z}(\omega) = z$ ).

Moreover, for any  $x \in \mathbb{R}^d$  and  $t \geq s$ , the random variable  $V_t^x$  is  $\mathcal{F}_{s,t}^L$ -measurable. The assertion is clear if  $t \leq u$ . Let us consider the case when  $t > u$ . First  $X_u^{s,x}$  is  $\mathcal{F}_{s,t}^L$ -measurable. Define  $F_{t,u}(z, \omega) = X_t^{u,z}(\omega)$ ,  $z \in \mathbb{R}^d$ ,  $\omega \in \Omega$ . The mapping  $F_{t,u}$  is clearly  $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_{s,t}^L$ -measurable on  $\mathbb{R}^d \times \Omega$  and  $F_{t,u}(\cdot, \omega)$  is continuous on  $\mathbb{R}^d$ , for any  $\omega \in \Omega$ , by (3.11). It follows that also the map:  $\omega \mapsto F_{t,u}(X_u^{s,x}(\omega), \omega)$  is  $\mathcal{F}_{s,t}^L$ -measurable.

It is clear that  $(V_t^x)$  solves (3.10) on  $\Omega'_s$  when  $s \leq t \leq u$  (recall (3.7)). Let us consider the case when  $t \geq u$ . According to (3.10) we know that on  $\Omega'_u$  we have

$$X_t^{u, X_u^{s,x}} = X_u^{s,x} + \int_u^t b(r, X_r^{u, X_u^{s,x}}) dr + L_t - L_u, \quad t \geq u. \quad (3.13)$$

Hence on  $\Omega'_u \cap \Omega'_s$  we have for  $t \geq u$

$$\begin{aligned} V_t^x &= X_t^{u, X_u^{s,x}} = x + \int_s^u b(r, X_r^{s,x}) dr + L_u - L_s \\ &+ \int_u^t b(r, X_r^{u, X_u^{s,x}}) dr + L_t - L_u = x + \int_s^t b(r, V_r^x) dr + L_t - L_s. \end{aligned}$$

It follows that  $(V_t^x)$  solves (3.10) on  $\Omega'_s \cap \Omega'_u$  when  $s \leq t \leq T$ . By Hypothesis 1 we infer that, for any  $x \in \mathbb{R}^d$ , on some almost sure event  $\Omega_{s,u,x}$  we have that  $V_t^x = X_t^{s,x}$ ,  $t \in [s, T]$ . In particular we get  $V_t^x = X_t^{s,x}$ ,  $t \in [u, T]$  and this proves (3.12) at least on an almost sure event  $\Omega_{s,u,x}$ .

To remove the dependence on  $x$  in the almost sure event, we note that the mapping:  $x \mapsto V_t^x(\omega)$  is continuous from  $\mathbb{R}^d$  into  $\mathbb{R}^d$ , for any  $\omega \in \Omega$ ,  $t \in [0, T]$  (see (3.11)). Arguing as in the final part of the proof of Lemma 3.2 we obtain that  $X_t^{s,x}(\omega) = V_t^x(\omega)$ , for  $t \in [u, T]$ ,  $x \in \mathbb{R}^d$  and  $\omega \in \Omega_{s,u} = \bigcap_{x \in \mathbb{Q}^d} \Omega_{s,u,x}$ . This proves (3.12).  $\blacksquare$

Following [26] page 169 (see also Problem 48 in [26]) we introduce the space  $C(\mathbb{R}^d; G_0)$  consisting of all continuous functions from  $\mathbb{R}^d$  into  $G_0 = C([0, T]; \mathbb{R}^d)$  endowed with the compact-open topology (or the topology of the uniform convergence on compact sets). This is a complete metric space endowed with the following metric:

$$d_0(f, g) = \sum_{N \geq 1} \frac{1}{2^N} \frac{\sup_{|x| \leq N} \|f(x) - g(x)\|_{G_0}}{1 + \sup_{|x| \leq N} \|f(x) - g(x)\|_{G_0}}, \quad f, g \in C(\mathbb{R}^d; G_0). \quad (3.14)$$

It is well-known that  $C(\mathbb{R}^d; G_0)$  is also *separable* (see, for instance, [16]; on the other hand  $C_b(\mathbb{R}^d; G_0)$  is not separable). We will also consider the following projections

$$\pi_x : C(\mathbb{R}^d; G_0) \rightarrow G_0, \quad \pi_x(f) = f(x) \in G_0, \quad x \in \mathbb{R}^d, \quad f \in C(\mathbb{R}^d; G_0) \quad (3.15)$$

(each  $\pi_x$  is a continuous map). According to Lemma 3.2 for any  $s \in [0, T]$  the random field  $(Y^{s,x})_{x \in \mathbb{R}^d}$  has continuous paths. It is not difficult to prove that, for any  $s \in [0, T]$ , the mapping:

$$\omega \mapsto Y^s(\omega) = Y^{s, \cdot}(\omega) \quad (3.16)$$

is measurable from  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in  $C(\mathbb{R}^d; G_0)$ . Indeed thanks to the separability of  $C(\mathbb{R}^d; G_0)$  to check the measurability it is enough to prove that counter-images of balls  $B_r(f_0) = \{f \in C(\mathbb{R}^d; G_0) : \sum_{N \geq 1} \frac{1}{2^N} \frac{\sup_{\{|x| \leq N, x \in \mathbb{Q}^d\}} \|f(x) - f_0(x)\|_{G_0}}{1 + \sup_{\{|x| \leq N, x \in \mathbb{Q}^d\}} \|f(x) - f_0(x)\|_{G_0}} < r\}$ ,  $r > 0$ ,  $f_0 \in C(\mathbb{R}^d; G_0)$ , are events in  $\Omega$ .

In the sequel we will set  $Y = (Y^s)_{s \in [0, T]}$  to denote the previous stochastic process with values in  $C(\mathbb{R}^d; G_0)$  and defined on  $(\Omega, \mathcal{F}, P)$ .

## 4 A version of the solution which is càdlàg with respect to the initial time $s$

In Theorem 4.4 we will prove the existence of a càdlàg modification  $Z$  of the process  $Y = (Y^s)_{s \in [0, T]}$  with values in  $C(\mathbb{R}^d; G_0)$  (cf. (3.16)). In particular  $Z$  is a modification of  $Y$  which is càdlàg in  $s$  uniformly in  $x$ , when  $x$  varies on compact sets of  $\mathbb{R}^d$ .

In Lemma 4.5 we will study important properties of  $Z$ . Before discussing on càdlàg modifications we recall a standard definition.

A process  $X = (X_t)_{t \in [0, T]}$  defined on  $(\Omega, \mathcal{F}, P)$  with values in a metric space  $(S, d)$  is *stochastically continuous (or continuous in probability)* if for any  $t_0 \in [0, T]$ ,  $X_t$  converges to  $X_{t_0}$  in probability (see [11] for more details).

Important results on càdlàg modifications for stochastic processes were given by Gikhman and Skorokhod (see Section III.4 in [11]). We will use a recent result given in Theorem 4.2 of [4]. In contrast with [11] the proof of this theorem does not require the separability of the stochastic process. It is stated in [4] for stochastic processes  $(X_t)$  when  $t \in [0, 1]$ . However a simple rescaling argument shows that it holds when  $t \in [0, T]$ , for any  $T > 0$ .

**Theorem 4.1.** ([4]) *Let  $X = (X_t)_{t \in [0, T]}$  be a stochastically continuous process defined on a complete probability space and with values in a complete metric space  $(S, d)$ . Let  $0 \leq s < t < u \leq T$  and define  $\Delta(s, t, u) = d(X_s, X_t) \wedge d(X_t, X_u)$ . A sufficient assumption in order that  $X$  has a modification with càdlàg paths is the following one: there exist non-negative real functions  $\delta$  and  $x_0$  ( $\delta$  is non-decreasing and continuous on  $[0, T]$ ,  $\delta(0) = 0$ , and  $x_0$  is decreasing and integrable on  $(0, T]$ ) such that the following conditions hold, for any  $0 \leq s < t < u \leq T$ ,  $M > 0$ ,*

$$E[\Delta(s, t, u) 1_{\Delta(s, t, u) \geq M}] \leq \delta(u - s) \int_0^{P(\Delta(s, t, u) \geq M)} x_0(r) dr, \quad (4.1)$$

$$\int_0^1 \left( u^{-1} \int_0^u x_0(r) dr \right) \frac{\delta(u)}{u} du < \infty. \quad (4.2)$$

The next result follows easily (cf. Section III.4 in [11]).

**Corollary 4.2.** *Let  $X = (X_t)_{t \in [0, T]}$  be a stochastically continuous process with values in a complete metric space  $(S, d)$ . A sufficient condition in order that  $X$  has a càdlàg modification is the following one: there exists  $q > 1/2$  and  $r > 0$  such that, for any  $0 \leq s < t < u \leq T$ , we have*

$$E[d(X_s, X_t)^q \cdot d(X_t, X_u)^q] \leq C|u - s|^{1+r}. \quad (4.3)$$

*Proof.* In order to apply Theorem 4.1 we introduce  $x_0(h) = \frac{2q-1}{2q} h^{-1/2q}$ ,  $h \in (0, T]$ . Let us fix  $0 \leq s < t < u$  and  $M > 0$ . Noting that for  $a, b \geq 0$  we have  $a \wedge b \leq \sqrt{a} \sqrt{b}$ . We find by the Hölder inequality

$$\begin{aligned} E[\Delta(s, t, u) 1_{\Delta(s, t, u) \geq M}] &\leq (E[\Delta(s, t, u)^{2q}])^{1/2q} (P(\Delta(s, t, u) \geq M))^{\frac{2q-1}{2q}} \\ &\leq E[d(X_s, X_t)^q \cdot d(X_t, X_u)^q]^{1/2q} \int_0^{P(\Delta(s, t, u) \geq M)} x_0(r) dr \\ &\leq C^{1/2q} |u - s|^{(1+r)/2q} \int_0^{P(\Delta(s, t, u) \geq M)} x_0(r) dr. \end{aligned}$$

Setting  $\delta(h) = h^{(1+r)/2q}$ ,  $h \in [0, T]$ , we see that  $\int_0^T \delta(u) u^{-1-\frac{1}{2q}} du < \infty$  is equivalent to (4.2); we get the assertion.  $\blacksquare$

We now prove the stochastic continuity of  $Y$ .

**Lemma 4.3.** Consider (2.2) with  $b \in L^\infty(0, T; C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d))$ ,  $\beta \in (0, 1]$ , and suppose that  $L$  and  $b$  satisfy Hypotheses 1 and 2. Then the process  $Y = (Y^s)$  with values in  $C(\mathbb{R}^d; G_0)$  (see (3.16)) is continuous in probability.

*Proof.* Let us fix  $s \in [0, T]$ . We have to prove that

$$\lim_{s' \rightarrow s} P\left(\sup_{|x| \leq N} \sup_{t \in [0, T]} |Y_t^{s,x} - Y_t^{s',x}| > r\right) = 0, \text{ for any } r > 0, N \geq 1. \quad (4.4)$$

Indeed this is equivalent to  $\lim_{s' \rightarrow s} P(d_0(Y^s, Y^{s'}) > r) = 0$ ,  $r > 0$ . To this purpose it is enough to check both the left and the right continuity in (4.4). Let us check the right continuity in  $s$  (assuming  $s \in [0, T]$ ). The proof of the left-continuity in  $s$  can be done in a similar way. Since  $C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d) \subset C_b^{0,\beta'}(\mathbb{R}^d; \mathbb{R}^d)$  for  $0 < \beta' \leq \beta \leq 1$  we may suppose that  $\beta$  is sufficiently small; we will assume (cf. Hypothesis 2)

$$\beta(2d+1) < 2d\theta. \quad (4.5)$$

Let  $(s_n) \subset ]s, T]$  with  $s_n \rightarrow s$ . We have to prove that for fixed  $N \geq 1$ ,  $\delta > 0$ ,

$$\lim_{n \rightarrow \infty} P\left(\sup_{|x| \leq N} \sup_{t \in [0, T]} |Y_t^{s,x} - Y_t^{s_n,x}| > \delta\right) = 0. \quad (4.6)$$

If we show that

$$E\left[\sup_{0 \leq t \leq T} \sup_{|x| \leq N} |Y_t^{s,x} - Y_t^{s_n,x}|\right] \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4.7)$$

then (4.6) follows. Let us fix  $n \geq 1$  and consider the random variable  $J_{t,x,n,s} = |Y_t^{s,x} - Y_t^{s_n,x}|$ . If  $t \leq s$  we find  $J_{t,x,n,s} = 0$ . If  $s \leq t \leq s_n$  then, for any  $x \in \mathbb{R}^d$ , on some almost sure event  $\Omega_{s,s_n}$  (independent of  $x$  and  $t$ ; see (3.7))

$$J_{t,x,n,s} = \left| \int_s^t b(r, Y_r^{s_n,x} + (L_r - L_s)) dr \right| \leq \|b\|_0 |t - s| \leq \|b\|_0 |s - s_n|.$$

Hence in order to get (4.7) we need to prove that

$$E\left[\sup_{s_n \leq t \leq T} \sup_{|x| \leq N} |Y_t^{s,x} - Y_t^{s_n,x}|\right] \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4.8)$$

Let  $t \geq s_n$ . We have on  $\Omega_{s,s_n}$

$$\begin{aligned} \sup_{|x| \leq N} |Y_t^{s,x} - Y_t^{s_n,x}| &\leq \sup_{|x| \leq N} \left| \int_s^t b(r, X_r^{s,x}) dr - \int_{s_n}^t b(r, X_r^{s_n,x}) dr \right| \\ &\leq 2|s - s_n| \|b\|_0 + \sup_{|x| \leq N} \int_{s_n}^t |b(r, X_r^{s,x}) - b(r, X_r^{s_n,x})| dr. \end{aligned} \quad (4.9)$$

By Lemma 3.3 on some almost sure event  $\Omega'_{s,s_n} \subset \Omega_{s,s_n}$  (independent of  $x$  and  $r$ ) we have for  $r \in [s_n, T]$

$$\begin{aligned} \sup_{|x| \leq N} |b(r, X_r^{s,x}) - b(r, X_r^{s_n,x})| &= \sup_{|x| \leq N} |b(r, X_r^{s_n, X_{s_n}^{s,x}}) - b(r, X_r^{s_n,x})| \\ &\leq [b]_{\beta,T} \sup_{|x| \leq N} \sup_{r \in [0, T]} |X_r^{s_n, X_{s_n}^{s,x}} - X_r^{s_n,x}|^\beta = [b]_{\beta,T} \sup_{|x| \leq N} \|Y^{s_n, X_{s_n}^{s,x}} - Y^{s_n,x}\|_{G_0}^\beta. \end{aligned}$$

By Lemma 3.2 with  $p = 4d$ , setting  $U_{s'} = U_{s',p}$ ,  $s' \in [0, T]$ , we get

$$\begin{aligned} & \sup_{|x| \leq N} |b(r, X_r^{s,x}) - b(r, X_r^{s_n,x})| \\ & \leq [b]_{\beta,T} [(|x| \vee |X_{s_n}^{s,x}|)^{\frac{2d+1}{4d}} \vee 1]^\beta U_{s_n}^\beta \sup_{|x| \leq N} |x - X_{s_n}^{s,x}|^{\beta/2}. \end{aligned} \quad (4.10)$$

Noting that, for  $|x| \leq N$ ,  $n \geq 1$ ,  $|X_{s_n}^{s,x}| \leq N + 2T\|b\|_0 + |L_{s_n} - L_s|$  we obtain on  $\Omega'_{s,s_n}$

$$\sup_{|x| \leq N} |b(r, X_r^{s,x}) - b(r, X_r^{s_n,x})| \leq [b]_{\beta,T} V_{s,s_n,N}^\beta \sup_{|x| \leq N} |x - X_{s_n}^{s,x}|^{\beta/2}, \quad (4.11)$$

$r \in [s_n, T]$ , where we have introduced the random variables

$$V_{s,s',N} = \left[ N^{\frac{2d+1}{4d}} + (2T\|b\|_0)^{\frac{2d+1}{4d}} + |L_{s'} - L_s|^{\frac{2d+1}{4d}} \right] U_{s'}, \quad (4.12)$$

$0 \leq s < s' \leq T$ . By Remark 2.1 and (4.5) we know that, for any  $n \geq 1$ ,

$$E[|L_{s_n} - L_s|^{\frac{\beta(2d+1)}{2d}}] = E[|L_{s_n-s}|^{\frac{\beta(2d+1)}{2d}}] \leq E[\sup_{s \in [0,T]} |L_s|^{\frac{\beta(2d+1)}{2d}}] < \infty,$$

since  $E[\sup_{r \in [0,T]} |L_r|^\theta] < \infty$ . Using also that  $\sup_{r \in [0,T]} E[U_{r,p}^{2\beta}] = k' < \infty$  (see (3.6)) we obtain by the Cauchy-Schwarz inequality

$$\sup_{0 \leq s < s' \leq T} E[V_{s,s',N}^\beta] = k_0 < \infty \quad (4.13)$$

( $k_0$  also depends on  $N$ ). Let us revert to (4.11). Since

$$|X_{s_n}^{s,x} - x| \leq \int_s^{s_n} |b(r, X_r^{s,x})| dr + |L_{s_n} - L_s| \leq \|b\|_0 |s - s_n| + |L_{s_n} - L_s|, \quad (4.14)$$

for any  $x \in \mathbb{R}^d$ ,  $n \geq 1$ , we obtain for  $r \in [s_n, T]$

$$\sup_{|x| \leq N} |b(r, X_r^{s,x}) - b(r, X_r^{s_n,x})| \leq [b]_{\beta,T} V_{s,s_n,N}^\beta (\|b\|_0 |s - s_n| + |L_{s_n} - L_s|)^{\beta/2} \quad (4.15)$$

and so (cf. (4.9))

$$\sup_{|x| \leq N} \int_{s_n}^t |b(r, X_r^{s,x}) - b(r, X_r^{s_n,x})| dr \leq T [b]_{\beta,T} V_{s,s_n,N}^\beta (\|b\|_0 |s - s_n| + |L_{s_n} - L_s|)^{\beta/2}.$$

Let us define the random variables  $Z_n = \|b\|_0 |s - s_n| + |L_{s_n} - L_s|$ . By the stochastic continuity of  $L$  we know that

$$\lim_{n \rightarrow \infty} P(Z_n > \delta) = 0, \quad \delta > 0. \quad (4.16)$$

Using (4.9) on an almost sure event  $\Omega'_{s,s_n}$ , for any  $\delta > 0$ , we have

$$\begin{aligned} & \sup_{s_n \leq t \leq T} \sup_{|x| \leq N} |Y_t^{s,x} - Y_t^{s_n,x}| \\ & \leq 2|s - s_n| \|b\|_0 + (1_{\{Z_n \leq \delta\}} + 1_{\{Z_n > \delta\}}) \cdot \sup_{|x| \leq N} \int_{s_n}^t |b(r, X_r^{s,x}) - b(r, X_r^{s_n,x})| dr \\ & \leq T 1_{\{Z_n \leq \delta\}} [b]_{\beta,T} V_{s,s_n,N}^\beta \delta^{\beta/2} + 2T \|b\|_0 1_{\{Z_n > \delta\}} + 2|s - s_n| \|b\|_0. \end{aligned}$$

Applying the expectation and using (4.13) we arrive at

$$\begin{aligned} & E\left[\sup_{s_n \leq t \leq T} \sup_{|x| \leq N} |Y_t^{s,x} - Y_t^{s_n,x}|\right] \\ & \leq 2|s - s_n| \|b\|_0 + k_0 T [b]_{\beta,T} \delta^{\beta/2} + 2T \|b\|_0 P(Z_n > \delta). \end{aligned}$$

Now, using (4.16), we obtain easily (4.8) and this completes the proof.  $\blacksquare$

In the next result we need the Lévy-Itô formula. To this purpose we recall the definition of Poisson random measure  $N$ :  $N((0, t] \times H) = \sum_{0 < s \leq t} 1_H(\Delta L_s)$  for any Borel set  $H$  in  $\mathbb{R}^d \setminus \{0\}$ ;  $\Delta L_s = L_s - L_{s-}$  denotes the jump size of  $L$  at time  $s > 0$ . The Lévy-Itô decomposition of the given Lévy process  $L$  on  $(\Omega, \mathcal{F}, P)$  with generating triplet  $(\nu, Q, 0)$  (see Section 19 in [28] or Theorem 2.4.16 in [1]) asserts that there exists a  $Q$ -Wiener process  $B = (B_t)$  on  $(\Omega, \mathcal{F}, P)$  independent of  $N$  with covariance matrix  $Q$  (cf. (2.1)) such that on some almost sure event  $\Omega'$  we have

$$L_t = A_t + B_t + C_t, \quad t \geq 0, \quad \text{where} \quad (4.17)$$

$$A_t = \int_0^t \int_{\{|x| \leq 1\}} x \tilde{N}(ds, dx), \quad C_t = \int_0^t \int_{\{|x| > 1\}} x N(ds, dx); \quad (4.18)$$

$\tilde{N}$  is the compensated Poisson measure (i.e.,  $\tilde{N}(dt, dx) = N(dt, dx) - dt\nu(dx)$ ).

**Theorem 4.4.** *Under the same assumptions of Lemma 4.3 consider the process  $Y = (Y^s)$  with values in  $C(\mathbb{R}^d; G_0)$  (see (3.16)). There exists a modification  $Z = (Z^s)$  of  $Y$  with càdlàg paths.*

*Proof.* To prove the assertion we will apply Corollary 4.2. We already know by Lemma 4.3 that  $Y$  is continuous in probability.

In the proof we will use the fact that  $\int_{\{|x| > 1\}} |x|^\theta \nu(dx) < \infty$  for some  $\theta \in (0, 1)$ . This is not restrictive according to Remark 2.1. We proceed in some steps.

*Step I.* We establish simple moment estimates for the Lévy process  $L$ , using the Itô-Lévy decomposition (4.18).

Using basic properties of the martingales  $(A_t)$  and  $(B_t)$  we obtain

$$E|B_t|^2 = C_Q t, \quad E|A_t|^2 = t \int_{\{|x| \leq 1\}} |x|^2 \nu(dx), \quad t \geq 0 \quad (4.19)$$

Now we concentrate on the compound Poisson process  $C = (C_t)$ ; on  $\Omega$  we have

$$|C_t|^\theta = \left| \sum_{0 < s \leq t} \Delta L_s 1_{\{|\Delta L_s| > 1\}} \right|^\theta \leq \sum_{0 < s \leq t} |\Delta L_s|^\theta 1_{\{|\Delta L_s| > 1\}},$$

since the random sum is finite for any  $\omega \in \Omega$  and  $\theta \leq 1$ . Let  $f_0(x) = 1_{\{|x| > 1\}}(x) |x|^\theta$ ,  $x \in \mathbb{R}^d$ ; using a well-know result (cf. pages 145 and 150 in [17] or Section 2.3.2 in [1]) we get

$$\begin{aligned} E\left[\sum_{0 < s \leq t} |\Delta L_s|^\theta 1_{\{|\Delta L_s| > 1\}}\right] &= E\left[\int_0^t \int_{\{|x| > 1\}} |x|^\theta N(ds, dx)\right] \\ &= \int_{\mathbb{R}^d} f_0(x) \nu(dx) = \int_{\{|x| > 1\}} |x|^\theta \nu(dx) \end{aligned}$$

and so

$$E|C_t|^\theta \leq t \int_{\{|x|>1\}} |x|^\theta \nu(dx) = c_0 t, \quad t \geq 0. \quad (4.20)$$

*Step II.* Let  $0 \leq s < s' \leq T$ . Similarly to the proof of Lemma 4.3 in this step we establish estimates for the random variable  $J_{t,x,s,s'} = |Y_t^{s,x} - Y_t^{s',x}|$ .

If  $t \leq s$  we have  $J_{t,x,s,s'} = 0$ ,  $x \in \mathbb{R}^d$ . If  $s \leq t \leq s'$  then, for any  $x \in \mathbb{R}^d$ , on some almost sure event  $\Omega_{s,s'}$  (independent of  $t$  and  $x$ ) we find

$$|Y_t^{s,x} - Y_t^{s',x}| \leq \|b\|_0 |t - s| \leq \|b\|_0 |s - s'|.$$

Let  $t \geq s'$  and  $N \geq 1$ . We have (cf. (4.9))

$$\sup_{|x| \leq N} |Y_t^{s,x} - Y_t^{s',x}| \leq 2|s - s'| \|b\|_0 + \sup_{|x| \leq N} \int_{s'}^t |b(r, X_r^{s,x}) - b(r, X_r^{s',x})| dr. \quad (4.21)$$

Moreover, there exists an almost sure event  $\Omega'_{s,s'} \subset \Omega_{s,s'}$  such that on  $\Omega'_{s,s'}$  we have for  $r \in [s', T]$

$$\begin{aligned} \sup_{|x| \leq N} |b(r, X_r^{s,x}) - b(r, X_r^{s',x})| &= \sup_{|x| \leq N} |b(r, X_r^{s',X_{s'}^{s,x}}) - b(r, X_r^{s',x})| \\ &\leq [b]_{\beta,T} \sup_{|x| \leq N} \|Y^{s',X_{s'}^{s,x}} - Y^{s',x}\|_{G_0}^\beta. \end{aligned}$$

Now we use Lemma 3.2 with  $p \geq 32d$  to be fixed and get, for any  $r \in [s', T]$  on  $\Omega'_{s,s'}$  (cf. (4.10) and (4.11))

$$\begin{aligned} &\sup_{|x| \leq N} |b(r, X_r^{s,x}) - b(r, X_r^{s',x})| \\ &\leq [b]_{\beta,T} [(|x| \vee |X_{s'}^{s,x}|)^{\frac{2d+1}{p}} \vee 1]^\beta U_{s',p}^\beta \sup_{|x| \leq N} |x - X_{s'}^{s,x}|^{\beta(1-\frac{2d}{p})} \end{aligned}$$

and so

$$\begin{aligned} \sup_{|x| \leq N} |b(r, X_r^{s,x}) - b(r, X_r^{s',x})| &\leq [b]_{\beta,T} V_{s,s',N,p}^\beta \sup_{|x| \leq N} |x - X_{s'}^{s,x}|^{\beta(1-\frac{2d}{p})}, \quad (4.22) \\ V_{s,s',N,p} &= \left[ N^{\frac{2d+1}{p}} + (2T\|b\|_0)^{\frac{2d+1}{p}} + |L_{s'} - L_s|^{\frac{2d+1}{p}} \right] U_{s',p}, \end{aligned}$$

Coming back to (4.21) we find for  $t \geq s'$  on  $\Omega'_{s,s'}$

$$\begin{aligned} &\sup_{s' \leq t \leq T} \sup_{|x| \leq N} |Y_t^{s,x} - Y_t^{s',x}| \leq 2|s - s'| \|b\|_0 \\ &+ T 1_{\left\{ \sup_{|x| \leq N} |X_{s'}^{s,x} - x| \leq c_0 |s - s'|^{1/8} \right\}} [b]_{\beta,T} V_{s,s',N,p}^\beta \sup_{|x| \leq N} |x - X_{s'}^{s,x}|^{\beta(1-\frac{2d}{p})} \\ &+ 2T \|b\|_0 1_{\left\{ \sup_{|x| \leq N} |X_{s'}^{s,x} - x| > c_0 |s - s'|^{1/8} \right\}}, \end{aligned}$$

with  $c_0 > 0$  such that  $c_0 \rho^{1/8} - \|b\|_0 \rho \geq \rho^{1/8}$ , for any  $\rho \in [0, T]$ . We obtain on  $\Omega'_{s,s'}$

$$\begin{aligned} &\sup_{|x| \leq N} \sup_{t \in [0, T]} |Y_t^{s,x} - Y_t^{s',x}| = \sup_{|x| \leq N} \|Y^{s,x} - Y^{s',x}\|_{G_0} \quad (4.23) \\ &\leq C_1 |s - s'| + C_1 V_{s,s',N,p}^\beta |s - s'|^{\frac{\beta}{8}(1-\frac{2d}{p})} + C_1 1_{\left\{ \sup_{|x| \leq N} |X_{s'}^{s,x} - x| > c_0 |s - s'|^{1/8} \right\}}, \end{aligned}$$

with  $C_1 = 2(T \vee 1)\|b\|_{\beta,T} c_0^\beta$ . Since, for any  $x \in \mathbb{R}^d$ ,  $|X_{s'}^{s,x} - x| \leq |s' - s|\|b\|_0 + |L_{s'} - L_s|$  and, moreover,  $c_0|s - s'|^{1/8} - \|b\|_0|s - s'| \geq |s - s'|^{1/8}$ , we find on  $\Omega'_{s,s'}$

$$\begin{aligned} & \sup_{|x| \leq N} \|Y^{s,x} - Y^{s',x}\|_{G_0} \\ & \leq C_1(|s - s'| + V_{s,s',N,p}^\beta |s - s'|^{\frac{\beta}{8}(1-\frac{2d}{p})} + 1_{\{|L_{s'} - L_s| > |s - s'|^{1/8}\}}). \end{aligned} \quad (4.24)$$

Note that  $C_1$  is independent of  $s$ ,  $s'$  and  $N$ .

*Step III.* Using (4.24) we provide an estimate for  $d_0(Y^s, Y^{s'})$  (cf. (3.14)) when  $0 \leq s < s' \leq T$ .

We have (see (4.22))

$$V_{s,s',N,p}^\beta \leq \left[ N^{\frac{\beta(2d+1)}{p}} + (2T\|b\|_0)^{\frac{\beta(2d+1)}{p}} + |L_{s'} - L_s|^{\frac{\beta(2d+1)}{p}} \right] U_{s',p}^\beta$$

and so

$$\begin{aligned} d_0(Y^s, Y^{s'}) &= \sum_{N \geq 1} \frac{1}{2^N} \frac{\sup_{|x| \leq N} \|Y^{s,x} - Y^{s',x}\|_{G_0}}{1 + \sup_{|x| \leq N} \|Y^{s,x} - Y^{s',x}\|_{G_0}} \\ &\leq C_1|s - s'| + C_1 1_{\{|L_{s'} - L_s| > |s - s'|^{1/8}\}} \\ &\quad + C_1 U_{s',p}^\beta |s - s'|^{\frac{\beta}{8}(1-\frac{2d}{p})} \sum_{N \geq 1} \frac{1}{2^N} \left[ N^{\frac{\beta(2d+1)}{p}} + (2T\|b\|_0)^{\frac{\beta(2d+1)}{p}} + |L_{s'} - L_s|^{\frac{\beta(2d+1)}{p}} \right] \\ &\leq C_3 \left( |s - s'| + 1_{\{|L_{s'} - L_s| > |s - s'|^{1/8}\}} + U_{s',p}^\beta |s - s'|^{\frac{\beta}{8}(1-\frac{2d}{p})} (1 + |L_{s'} - L_s|^{\frac{\beta(2d+1)}{p}}) \right), \end{aligned}$$

where  $C_3 = C_3(\beta, T, \|b\|_{\beta,T}, d, p) > 0$ . Recall that  $p \geq 32d$  has to be fixed.

*Step IV.* Let now  $0 \leq s_1 < s_2 < s_3 \leq T$  and set

$$\rho = s_3 - s_1.$$

We will apply Corollary 4.2 with  $q = 8/\beta$ . Let us fix  $p \geq 32d$  (i.e.,  $1 - \frac{2d}{p} \geq 15/16$ ) such that  $\frac{8(2d+1)}{p} < \frac{\theta}{4}$  and introduce the random variable

$$Z = 1 + \sup_{s \in [0, T]} |L_s|^{\frac{8(2d+1)}{p}},$$

Clearly we have that  $|L_{s'} - L_s|^{\frac{8(2d+1)}{p}} \leq 2Z$ ,  $0 \leq s < s' \leq T$ . Moreover by Remark 2.1 we know that  $E[Z^4] < \infty$ . Using Step III and the previous estimates we will check condition (4.3). In the sequel we denote by  $C_k$  or  $c_k$  positive constants which may depend on  $\beta, T, \|b\|_{\beta,T}, \theta$  and  $d$  but are independent of  $s_1, s_2$  and  $s_3$ . We have

$$\begin{aligned} \Gamma &= E \left[ \left( d_0(Y^{s_1}, Y^{s_2}) \cdot d_0(Y^{s_2}, Y^{s_3}) \right)^{8/\beta} \right] \\ &\leq C_4 E \left[ \left( |s_3 - s_1|^{8/\beta} + 1_{\{|L_{s_2} - L_{s_1}| > |s_2 - s_1|^{1/8}\}} + Z U_{s_2,p}^8 |s_3 - s_1|^{1-\frac{2d}{p}} \right) \right. \\ &\quad \cdot \left. \left( |s_3 - s_1|^{8/\beta} + 1_{\{|L_{s_3} - L_{s_2}| > |s_3 - s_2|^{1/8}\}} + Z U_{s_3,p}^8 |s_3 - s_1|^{1-\frac{2d}{p}} \right) \right]. \end{aligned}$$



We denote by  $c_2 \geq 1$  a constant such that  $t^{8/\beta} \leq c_2 t^{1-\frac{2d}{p}}$ ,  $t \in [0, T]$ . We obtain ( $\rho = s_3 - s_1$ )

$$\begin{aligned} \Gamma &\leq c_2^2 C_4 E \left[ \left( \rho^{1-\frac{2d}{p}} + 1_{\{|L_{s_2}-L_{s_1}| > |s_2-s_1|^{1/8}\}} + Z U_{s_2,p}^8 \rho^{1-\frac{2d}{p}} \right) \right. \\ &\quad \cdot \left. \left( \rho^{1-\frac{2d}{p}} + 1_{\{|L_{s_3}-L_{s_2}| > |s_3-s_2|^{1/8}\}} + Z U_{s_3,p}^8 \rho^{1-\frac{2d}{p}} \right) \right] \\ &\leq C_5 (\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4), \end{aligned}$$

where  $\Gamma_1 = E[1_{\{|L_{s_2}-L_{s_1}| > |s_2-s_1|^{1/8}\}} \cdot 1_{\{|L_{s_3}-L_{s_2}| > |s_3-s_2|^{1/8}\}}]$ ,

$$\Gamma_2 = \rho^{1-\frac{2d}{p}} [P(|L_{s_3} - L_{s_2}| > |s_3 - s_2|^{1/8} + P(|L_{s_2} - L_{s_1}| > |s_2 - s_1|^{1/8})],$$

$$\Gamma_3 = \rho^{1-\frac{2d}{p}} E[1_{\{|L_{s_3}-L_{s_2}| > |s_3-s_2|^{1/8}\}} Z U_{s_2,p}^8 + 1_{\{|L_{s_2}-L_{s_1}| > |s_2-s_1|^{1/8}\}} Z U_{s_3,p}^8],$$

$$\Gamma_4 = \rho^{2(1-\frac{2d}{p})} + \rho^{2(1-\frac{2d}{p})} E[Z U_{s_2,p}^8 + Z U_{s_3,p}^8 + Z^2 U_{s_2,p}^8 U_{s_3,p}^8].$$

It is not difficult to treat  $\Gamma_4$ . Indeed we can use the Cauchy-Schwarz inequality and

$$\sup_{s \in [0, T]} E[U_{s,p}^p] = k' < \infty \quad (4.25)$$

(see (3.6)) in order to control the expectation in  $\Gamma_4$ . For instance, we have

$$E[Z^2 U_{s_2,p}^8 U_{s_3,p}^8] \leq E[Z^4]^{1/2} \left( \sup_{s \in [0, T]} E[U_{s,p}^{32}] \right)^{1/2} < \infty, \quad (4.26)$$

since  $E[Z^4] < \infty$  and  $p \geq 32d$ . We obtain

$$\Gamma_4 \leq C_6 \rho^{2(1-\frac{2d}{p})} = C_6 |s_3 - s_1|^{2(1-\frac{2d}{p})} \leq C_6 |s_3 - s_1|^{30/16}. \quad (4.27)$$

To estimates the other terms we need to control  $P(|L_s| > |s|^{1/8})$ ,  $s \geq 0$ . To this purpose we use Step I. We have

$$P(|L_s| > s^{1/8}) \leq P(|B_s| > s^{1/8}/3) + P(|A_s| > s^{1/8}/3) + P(|C_s| > s^{1/8}/3).$$

By Chebychev inequality we get for  $s \geq 0$

$$P(|L_s| > s^{1/8}) \leq \frac{9}{s^{1/4}} E[|B_s|^2 + |A_s|^2] + \frac{3^\theta}{s^{\theta/8}} E[|C_s|^\theta] \leq c_3 (s^{3/4} + s^{1-\frac{\theta}{8}}). \quad (4.28)$$

Using (4.28) and (4.25) we can estimate  $\Gamma_2$  and  $\Gamma_3$ . For instance, since the increments of  $L$  are independent and stationary, we find

$$\begin{aligned} \Gamma_2 &\leq \rho^{1-\frac{2d}{p}} [P(|L_{s_3-s_2}| > |s_3 - s_2|^{1/8}) + P(|L_{s_2-s_1}| > |s_2 - s_1|^{1/8})] \\ &\leq 2c_3 \rho^{1-\frac{2d}{p}} (\rho^{3/4} + \rho^{1-\frac{\theta}{8}}). \end{aligned}$$

We can proceed similary for  $\Gamma_3$  (see also (4.26)):

$$\begin{aligned} \Gamma_3 &\leq \rho^{1-\frac{2d}{p}} (E[Z^4])^{1/4} \left( \sup_{s \in [0, T]} E[U_{s,p}^{32}] \right)^{1/4} \left[ (P(|L_{s_3-s_2}| > |s_3 - s_2|^{1/8})^{1/2} \right. \\ &\quad \left. + (P(|L_{s_2-s_1}| > |s_2 - s_1|^{1/8}))^{1/2} \right] \leq C_8 \rho^{1-\frac{2d}{p}} (\rho^{3/8} + \rho^{\frac{1}{2}(1-\frac{\theta}{8})}). \end{aligned}$$

Note that  $(1 - \frac{2d}{p}) + 3/8 > 5/4$  and  $(1 - \frac{2d}{p}) + \frac{1}{2}(1 - \frac{\theta}{8}) > 5/4$ . We get

$$\Gamma_2 + \Gamma_3 \leq C_9 \rho^{\frac{5}{4}} = C_9 |s_3 - s_1|^{5/4}. \quad (4.29)$$

Finally we consider

$$\begin{aligned} \Gamma_1 &\leq (P(|L_{s_3-s_2}| > |s_3 - s_2|^{1/8}) \cdot (P(|L_{s_2-s_1}| > |s_2 - s_1|^{1/8})) \\ &\leq 2c_3(\rho^{3/2} + \rho^{2(1-\frac{\theta}{8})}) \leq c_4 |s_3 - s_1|^{3/2}. \end{aligned} \quad (4.30)$$

Collecting together estimates (4.27), (4.29) and (4.30) we arrive at

$$E \left[ \left( d_0(Y^{s_1}, Y^{s_2}) \cdot d_0(Y^{s_2}, Y^{s_3}) \right)^{8/\beta} \right] \leq C_0 |s_3 - s_1|^{5/4}$$

and this finishes the proof.  $\blacksquare$

Taking into account Theorem 4.4 and using the projections  $\pi_x$  (see (3.15)), in the sequel we write, for  $x \in \mathbb{R}^d$ ,  $s, t \in [0, T]$ ,

$$Z^s = (Z^{s,x})_{x \in \mathbb{R}^d}, \quad \text{with } \pi_x(Z^s) = Z^{s,x} \in G_0. \quad (4.31)$$

Recall that on some almost sure event  $\Omega_s$ ,  $Y^{s,x} = Z^{s,x}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  (cf. (3.16)).

**Lemma 4.5.** *Under the same assumptions of Lemma 4.3 consider the càdlàg process  $Z$  with values in  $C(\mathbb{R}^d; G_0)$  of Theorem 4.4. The following statements hold:*

(i) *There exists an almost sure event  $\Omega_1$  (independent of  $s, t$  and  $x$ ) such that for any  $\omega \in \Omega_1$ , we have that  $t \mapsto L_t(\omega)$  is càdlàg,  $L_0(\omega) = 0$  and  $s \mapsto Z^s(\omega)$  is càdlàg; further, for any  $\omega \in \Omega_1$ ,*

$$Z_t^{s,x}(\omega) = x + \int_s^t b(r, Z_r^{s,x}(\omega) + L_r(\omega) - L_s(\omega)) dr, \quad s, t \in [0, T], \quad s \leq t, \quad x \in \mathbb{R}^d.$$

Moreover, for  $s \leq t$ , the r.v.  $Z_t^{s,x}$  is  $\mathcal{F}_{s,t}^L$ -measurable (if  $t \leq s$ ,  $Z_t^{s,x} = x$ ).

(ii) *There exists an almost sure event  $\Omega_2$  and a  $\mathcal{B}([0, T]) \times \mathcal{F}$ -measurable function  $V_n : [0, T] \times \Omega \rightarrow [0, \infty]$ , such that  $\int_0^T V_n(s, \omega) ds < \infty$ , for any integer  $n > 2d$ ,  $\omega \in \Omega_2$ , and, further, the following inequality holds on  $\Omega_2$*

$$\sup_{t \in [0, T]} |Z_t^{s,x} - Z_t^{s,y}| \leq |x - y|^{\frac{n-2d}{n}} [(|x| \vee |y|)^{\frac{2d+1}{n}} \vee 1] V_n(s, \cdot), \quad x, y \in \mathbb{R}^d, \quad s \in [0, T]. \quad (4.32)$$

(iii) *There exists an almost sure event  $\Omega_3$  such that for any  $\omega \in \Omega_3$  we have*

$$Z_t^{s,x}(\omega) + L_u(\omega) - L_s(\omega) = Z_t^{u, Z_u^{s,x}(\omega) + L_u(\omega) - L_s(\omega)}(\omega), \quad (4.33)$$

for any  $s, u, t \in [0, T]$ ,  $0 \leq s < u \leq T$ ,  $x \in \mathbb{R}^d$ .

*Proof.* (i) On some almost sure event  $\Omega'_s$  (independent of  $t$  and  $x$ ) we know that  $(Y_t^{s,x})$  verifies the SDE (3.7) for any  $x \in \mathbb{R}^d$  and  $t \in [s, T]$ . Moreover  $Y_t^{s,x} = x$ ,  $t < s$ .

On the other hand on some almost sure event  $\Omega_s$  we have  $Y^{s,x} = \pi_x(Y^s) = \pi_x(Z^s)$ , for any  $x \in \mathbb{R}^d$ , see (4.31). Using  $(Z^s)$ , we can rewrite (3.7) on the event  $\Omega_1 = \bigcap_{r \in \mathbb{Q} \cap [0, T]} (\Omega'_r \cap \Omega_r)$  as follows:

$$[\pi_x(Z^s)]_t = x + \int_s^t b(r, [\pi_x(Z^s)]_r + (L_r - L_s)) dr, \quad (4.34)$$

for any  $s \in \mathbb{Q} \cap [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$ .

Note that by Theorem 4.4 for any  $\omega \in \Omega$  and any sequence  $s_n \rightarrow s^+$  we have  $d_0(Z^s(\omega), Z^{s_n}(\omega)) \rightarrow 0$  as  $n \rightarrow \infty$ . Take now  $s \in [0, T]$  and let  $(s_n) \subset \mathbb{Q} \cap [0, T]$  be a sequence monotonically decreasing to  $s$ . By the dominated convergence theorem and the right-continuity of  $L$  we have on  $\Omega_1$ , for any  $t > s$ ,  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} [\pi_x(Z^s)]_t &= \lim_{n \rightarrow \infty} [\pi_x(Z^{s_n})]_t = x + \lim_{n \rightarrow \infty} \int_s^t 1_{\{r > s_n\}} b(r, [\pi_x(Z^{s_n})]_r + (L_r - L_{s_n})) dr \\ &= x + \int_s^t b(r, [\pi_x(Z^s)]_r + (L_r - L_s)) dr \end{aligned}$$

and we get the assertion.

(ii) Since on  $\Omega$  we have  $Y^{s,x} = \pi_x(Y^s)$  we obtain by (2.6) and (3.9), for any  $p \geq 2$ ,

$$\begin{aligned} \sup_{s \in [0, T]} E[ \sup_{s \leq t \leq T} |X_t^{s,x} - X_t^{s,y}|^p ] &= \sup_{s \in [0, T]} E[ \sup_{0 \leq t \leq T} |Y_t^{s,x} - Y_t^{s,y}|^p ] \\ &= \sup_{s \in [0, T]} E[ \|\pi_x(Z^s) - \pi_y(Z^s)\|_{G_0}^p ] \leq C(T) |x - y|^p, \quad x, y \in \mathbb{R}^d. \end{aligned} \quad (4.35)$$

Let  $s \in [0, T]$  and consider the random field  $(\pi_x(Z^s))_{x \in \mathbb{R}^d}$  with values in  $G_0$ . Applying Theorem 3.1 with  $\psi(x, \omega) = \pi_x(Z^s)(\omega)$  we obtain from (4.35) for  $p > 2d$  similarly to (3.5): there exists a  $V_p(s, \omega) \in [0, \infty]$  such that, for any  $\omega \in \Omega$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,

$$\|\pi_x(Z^s)(\omega) - \pi_y(Z^s)(\omega)\|_{G_0} \leq [(|x| \vee |y|)^{\frac{2d+1}{p}} \vee 1] V_p(s, \omega) |x - y|^{1-2d/p}, \quad (4.36)$$

with  $V_p(s, \omega) = \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{\|\pi_x(Z^s)(\omega) - \pi_y(Z^s)(\omega)\|_{G_0}}{|x - y|} \right)^p f(x) f(y) dx dy \right)^{1/p}$ ,  $\omega \in \Omega$ ,  $s \in [0, T]$  ( $f$  is defined in Theorem 3.1). Since the map:  $(s, x, \omega) \mapsto \pi_x(Z^s)(\omega)$  is  $\mathcal{B}([0, T] \times \mathbb{R}^d) \times \mathcal{F}$ -measurable with values in  $G_0$ , it follows that the real map:

$$(s, x, y, \omega) \mapsto \|\pi_x(Z^s)(\omega) - \pi_y(Z^s)(\omega)\|_{G_0} |x - y|^{-1} 1_{\{x \neq y\}}$$

is  $\mathcal{B}([0, T] \times \mathbb{R}^{2d}) \times \mathcal{F}$ -measurable. By the Fubini theorem we deduce that also  $V_p : [0, T] \times \Omega \rightarrow [0, \infty]$  is  $\mathcal{B}([0, T]) \times \mathcal{F}$ -measurable. Hence we can consider the random variable  $\omega \mapsto \int_0^T V_p(s, \omega) ds$  (with values in  $[0, \infty]$ ). Since, with the same constant  $C(T)$  appearing in (2.6),

$$\sup_{s \in [0, T]} E[|V_p(s, \cdot)|^p] \leq C(d) \cdot C(T), \quad (4.37)$$

we find  $E \left[ \left( \int_0^T V_p(s, \cdot) ds \right)^p \right] \leq T^{p-1} \int_0^T E[(V_p(s, \cdot))^p] ds \leq T^{2p-1} c(d) C(T) < \infty$ . It follows that, for any  $p > 2d$ , there exists an almost sure event  $\Omega_p$  such that

$$\int_0^T V_p(s, \omega) ds < \infty, \quad \omega \in \Omega_p. \quad (4.38)$$

Let  $p = n$ . We find, for any  $n > 2d$ ,  $\int_0^T V_n(s, \omega) ds < \infty$ , when  $\omega \in \Omega_2 = \bigcap_{n > 2d} \Omega_n$ .

Writing (4.36) for  $\omega \in \Omega_2$  and  $n > 2d$  we find the assertion.

(iii) First note that the statement of Lemma 3.3 can be rewritten in term of the process  $Y^{s,x}$  (see (3.9)) as follows: for any  $0 \leq s < u \leq T$  there exists an almost sure event  $\Omega_{s,u}$  (independent of  $t$  and  $x$ ) such that, for any  $\omega \in \Omega_{s,u}$ , we have

$$Y_t^{s,x}(\omega) + L_u(\omega) - L_s(\omega) = Y_t^{u, Y_u^{s,x}(\omega) + L_u(\omega) - L_s(\omega)}(\omega), \quad \text{for } t \in [u, T], x \in \mathbb{R}^d. \quad (4.39)$$

Since  $(Z^s)$  is a modification of  $(Y^s)$  (see Theorem 4.4) we know that on some almost sure event  $\Omega''_{s,u} \subset \Omega_{s,u}$  identity (4.39) holds when  $(Y^{s,x})$  is replaced by  $(Z^{s,x})$ .

Let us fix  $u \in (0, T]$ . We know that (4.39) holds for  $(Z^{s,x})$  when  $t \in [u, T]$ ,  $x \in \mathbb{R}^d$  and  $s \in [0, u) \cap \mathbb{Q}$  if  $\omega \in \Omega_u = \cap_{s \in [0, u) \cap \mathbb{Q}} (\Omega''_{s,u} \cap \Omega_1)$ . Using that  $(Z^{s,x})$  with values in  $G_0$  is in particular right-continuous in  $s$ , uniformly in  $x$ , when  $x$  varies in compact sets of  $\mathbb{R}^d$ , it is easy to check that (4.33) holds, for any  $0 \leq s < u \leq T$ ,  $x \in \mathbb{R}^d$ ,  $t \in [u, T]$ , when  $\omega \in \Omega_u$ .

Let us define  $\Omega_3 = \bigcap_{u \in \mathbb{Q} \cap [0, T)} \Omega_u$ ; fix any  $s, u_0 \in [0, T]$ ,  $x \in \mathbb{R}^d$ , with  $0 \leq s < u_0 \leq T$ ; we consider  $\omega \in \Omega_3$  and prove that (4.33) holds for any  $t \in [u_0, T]$ .

If  $t = u_0$  the assertion holds. Let us suppose that  $t \in (u_0, T]$ . We can find a sequence  $(u_j) \in (u_0, t) \cap \mathbb{Q}$  such that  $u_j \rightarrow u_0^+$ . Since for any  $j \geq 1$  we have

$$Z_t^{s,x}(\omega) + L_{u_j}(\omega) - L_s(\omega) = Z_t^{u_j, Z_{u_j}^{s,x}(\omega) + L_{u_j}(\omega) - L_s(\omega)}(\omega), \quad (4.40)$$

we can pass to the limit as  $j \rightarrow \infty$  in both sides of the previous formula (taking also into account that  $Z_{u_j}^{s,x}(\omega) + L_{u_j}(\omega) - L_s(\omega)$  belongs to a compact set  $K_{x,s,\omega} \subset \mathbb{R}^d$  for any  $j \geq 1$ ) and find that (4.40) holds when  $u_j$  is replaced by  $u_0$ . The proof of (4.33) is complete.  $\blacksquare$

## 5 A Davie's type uniqueness result

Assertion (v) of the next theorem gives a Davie's type uniqueness result for SDE (1.1). The other assertions collect results of Section 4 (see in particular Theorem 4.4 and Lemma 4.5). These are used to prove the uniqueness property (v). We refer to Corollaries 5.4 and 5.5 for the case when  $b(t, \cdot)$  is only locally Hölder continuous.

We stress that all the next statements (i)-(v) hold when  $\omega$  belongs to an almost sure event  $\Omega'$  (independent of  $s, t \in [0, T]$  and  $x \in \mathbb{R}^d$ ).

**Theorem 5.1.** *Let us consider the SDE (1.1) with  $b \in L^\infty(0, T; C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d))$ ,  $\beta \in (0, 1]$ , and suppose that  $L$  and  $b$  satisfy Hypotheses 1 and 2. Then there exists a function  $\phi(s, t, x, \omega)$ ,*

$$\phi : [0, T] \times [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d, \quad (5.1)$$

*which is  $\mathcal{B}([0, T] \times [0, T] \times \mathbb{R}^d) \times \mathcal{F}$ -measurable and such that  $(\phi(s, t, x, \cdot))_{t \in [0, T]}$  is a strong solution of (1.1) starting from  $x$  at time  $s$ . Moreover, there exists an almost sure event  $\Omega'$  such that the following assertions hold for any  $\omega \in \Omega'$ .*

(i) *For any  $x \in \mathbb{R}^d$ , the mapping:  $s \mapsto \phi(s, t, x, \omega)$  is càdlàg on  $[0, T]$  (uniformly in  $t$  and  $x$ ), i.e., let  $s \in (0, T)$  and consider sequences  $(s_k)$  and  $(r_n)$  such that  $s_k \rightarrow s^-$  and  $r_n \rightarrow s^+$ ; we have, for any  $M > 0$ ,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{|x| \leq M} \sup_{t \in [0, T]} |\phi(r_n, t, x, \omega) - \phi(s, t, x, \omega)| &= 0, \\ \lim_{k \rightarrow \infty} \sup_{|x| \leq M} \sup_{t \in [0, T]} |\phi(s_k, t, x, \omega) - \phi(s, t, x, \omega)| &= 0 \end{aligned} \quad (5.2)$$

*(similar conditions hold when  $s = 0$  and  $s = T$ ).*

(ii) *For any  $x \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $\phi(s, t, x, \omega) = x$  if  $0 \leq t \leq s$ , and*

$$\phi(s, t, x, \omega) = x + \int_s^t b(r, \phi(s, r, x, \omega)) dr + L_t(\omega) - L_s(\omega), \quad t \in [s, T]. \quad (5.3)$$

(iii) For any  $s \in [0, T]$ , the function  $x \mapsto \phi(s, t, x, \omega)$  is continuous in  $x$  uniformly in  $t$ . Moreover, for any integer  $n > 2d$ , there exists a  $\mathcal{B}([0, T]) \times \mathcal{F}$ -measurable function  $V_n : [0, T] \times \Omega \rightarrow [0, \infty]$  such that  $\int_0^T V_n(s, \omega) ds < \infty$  and

$$\begin{aligned} & \sup_{t \in [0, T]} |\phi(s, t, x, \omega) - \phi(s, t, y, \omega)| \\ & \leq V_n(s, \omega) |x - y|^{\frac{n-2d}{n}} [(|x| \vee |y|)^{\frac{2d+1}{n}} \vee 1], \quad x, y \in \mathbb{R}^d, n > 2d, s \in [0, T]. \end{aligned} \quad (5.4)$$

(iv) For any  $0 \leq s < r \leq t \leq T$ ,  $x \in \mathbb{R}^d$ , we have

$$\phi(s, t, x, \omega) = \phi(r, t, \phi(s, r, x, \omega), \omega). \quad (5.5)$$

(v) Let  $s_0 \in [0, T]$ ,  $\tau = \tau(\omega) \in (s_0, T]$  and  $x \in \mathbb{R}^d$ . If a measurable function  $g : [s_0, \tau] \rightarrow \mathbb{R}^d$  solves the integral equation

$$g(t) = x + \int_{s_0}^t b(r, g(r)) dr + L_t(\omega) - L_{s_0}(\omega), \quad t \in [s_0, \tau], \quad (5.6)$$

then we have  $g(r) = \phi(s_0, r, x, \omega)$ , for  $r \in [s_0, \tau]$ .

*Proof.* Let us consider the process  $Z = (Z^s)_{s \in [0, T]}$  of Theorem 4.4 with values in  $C(\mathbb{R}^d; G_0)$ . Recall the notation  $Z_t^{s, x} = \pi_x(Z^s)(t)$  (see (3.15)). We define for  $\omega \in \Omega$ ,  $s, t \in [0, T]$ ,  $x \in \mathbb{R}^d$ :

$$\phi(s, t, x, \omega) = Z_t^{s, x}(\omega) + L_t(\omega) - L_s(\omega), \quad \text{if } s \leq t, \quad (5.7)$$

and  $\phi(s, t, x, \omega) = x$  if  $s > t$ . The fact that, for any  $0 \leq s < t \leq T$ ,  $x \in \mathbb{R}^d$ , the random variable  $\phi(s, t, x, \cdot)$  is  $\mathcal{F}_{s, t}^L$ -measurable follow from Theorem 4.4 and (i) in Lemma 4.5. We also define

$$\Omega' = \Omega_1 \cap \Omega_2 \cap \Omega_3,$$

where the almost sure events  $\Omega_k$ ,  $k = 1, 2, 3$ , are considered in Lemma 4.5.

Assertions **(i)**, **(ii)**, **(iii)**, **(iv)** follow directly from Theorem 4.4 and Lemma 4.5. More precisely, (i) and (ii) follow from the first assertion of Lemma 4.5 since  $(Z^s)$  takes values in  $C(\mathbb{R}^d; G_0)$  with càdlàg paths. Assertions (iii) and (iv) follow respectively from the second and third assertion of Lemma 4.5.

**(v)** Let  $\omega \in \Omega'$  be fixed and let  $g : [s_0, \tau] \rightarrow \mathbb{R}^d$  be a solution to the integral equation (5.6) corresponding to  $\omega$ . Let us fix  $t \in (s_0, \tau)$ .

We introduce an auxiliary function  $f : [s_0, t] \rightarrow \mathbb{R}^d$  which is similar to the one used in proof of Theorem 3.1 in [30],

$$f(s) = \phi(s, t, g(s), \omega), \quad s \in [s_0, t]. \quad (5.8)$$

We will show that  $f$  is constant on  $[s_0, t]$ . Once this is proved we can deduce that  $f(t) = f(s_0)$  and so we find  $g(t) = \phi(s_0, t, x, \omega)$  which shows the assertion since  $t$  is arbitrary. In the sequel we proceed in three steps.

*Step I.* We establish some estimates for  $|g(r) - \phi(u, r, g(u), \omega)|$  when  $s_0 \leq u \leq r \leq t$ .

Since

$$g(r) = x + \int_{s_0}^u b(p, g(p)) dp + (L_u(\omega) - L_{s_0}(\omega)) + \int_u^r b(p, g(p)) dp + (L_r(\omega) - L_u(\omega)),$$

we obtain

$$\begin{aligned}
|g(r) - \phi(u, r, g(u), \omega)| &\leq \left| g(u) + \int_u^r b(p, g(p)) dp + (L_r(\omega) - L_u(\omega)) - g(u) \right. \\
&\quad \left. - \int_u^r b(p, \phi(u, p, g(u), \omega)) dp - (L_r(\omega) - L_u(\omega)) \right| \\
&\leq \int_u^r |b(p, g(p)) - b(p, \phi(u, p, g(u), \omega))| dp \leq 2\|b\|_0 |r - u|.
\end{aligned}$$

Now using the Hölder continuity of  $b$ :

$$\begin{aligned}
|g(r) - \phi(u, r, g(u), \omega)| &\leq \int_u^r |b(p, g(p)) - b(p, \phi(u, p, g(u), \omega))| dp \quad (5.9) \\
&\leq [b]_{\beta, T} \int_u^r |g(p) - \phi(u, p, g(u), \omega)|^\beta dp \\
&\leq (2\|b\|_0)^\beta [b]_{\beta, T} \int_u^r |p - u|^\beta dp \leq (2\|b\|_0)^\beta [b]_{\beta, T} |r - u|^{1+\beta}.
\end{aligned}$$

*II Step.* We prove that  $f$  defined in (5.8) is continuous on  $[s_0, t]$ .

We first show that it is right-continuous on  $[s_0, t)$ . Let us fix  $s \in [s_0, t)$  and consider a sequence  $(s_n)$  such that  $s_n \rightarrow s^+$ . We prove that  $f(s_n) \rightarrow f(s)$  as  $n \rightarrow \infty$ . Note that  $|g(r)| \leq M_0$ ,  $r \in [s_0, \tau)$ , where  $M_0 = |x| + T\|b\|_0 + C(\omega)$ . We have

$$\begin{aligned}
|f(s_n) - f(s)| &\leq |\phi(s_n, t, g(s_n), \omega) - \phi(s, t, g(s_n), \omega)| \\
&\quad + |\phi(s, t, g(s_n), \omega) - \phi(s, t, g(s), \omega)| \leq J_n + I_n,
\end{aligned}$$

where  $I_n = |\phi(s, t, g(s_n), \omega) - \phi(s, t, g(s), \omega)|$  and

$$J_n = \sup_{|x| \leq M_0} \sup_{t \in [0, T]} |\phi(s_n, t, x, \omega) - \phi(s, t, x, \omega)|.$$

Since  $g(s_n) \rightarrow g(s)$  by the right continuity of  $g$  we obtain that  $\lim_{n \rightarrow \infty} I_n = 0$  thanks to (5.4). Moreover  $\lim_{n \rightarrow \infty} J_n = 0$  thanks to (5.2).

Let us show that  $f$  is left-continuous on  $(s_0, t]$ . We fix  $s \in (s_0, t]$  and consider a sequence  $(s_k) \subset (s_0, s)$  such that  $s_k \rightarrow s$ . We prove that  $f(s_k) \rightarrow f(s)$  as  $k \rightarrow \infty$ . Using the flow property (iv) we find

$$\begin{aligned}
|f(s_k) - f(s)| &= |\phi(s_k, t, g(s_k), \omega) - \phi(s, t, g(s), \omega)| \\
&= |\phi(s, t, \phi(s_k, s, g(s_k), \omega), \omega) - \phi(s, t, g(s), \omega)|.
\end{aligned}$$

By I Step we know that

$$|\phi(s_k, s, g(s_k), \omega) - g(s)| \leq 2\|b\|_0 |s_k - s| \quad (5.10)$$

which tends to 0 as  $k \rightarrow \infty$ . Using (5.10) and the continuity property (iii) we obtain the claim since

$$\lim_{k \rightarrow \infty} |\phi(s, t, \phi(s_k, s, g(s_k), \omega), \omega) - \phi(s, t, g(s), \omega)| = 0.$$

*III Step.* We prove that  $f$  is constant on  $[s_0, t]$ .

We will use the following well known lemma (see, for instance, pages 239-240 in [36]): *Let  $S$  be a real Banach space and consider a continuous mapping  $F : [a, b] \subset \mathbb{R} \rightarrow S$ ,  $b > a$ . Suppose that for any  $h \in (a, b]$  there exists the left derivative*

$$\frac{d^- F}{dh}(h) = \lim_{h' \rightarrow h^-} \frac{F(h') - F(h)}{h' - h} \quad (5.11)$$

*and this derivative is identically zero on  $(a, b]$ . Then  $F$  is constant.*

Note that by considering continuous linear functionals on  $S$  one may reduce the proof of the lemma to the one of a real analysis result.

To apply the previous lemma with  $[s_0, t] = [a, b]$  we first extend our function  $f$  to  $[s_0, \infty)$  by setting  $f(r) = f(t)$  for  $r \geq t$ . Then set  $S = L^1([0, t]; \mathbb{R}^d)$  and define  $F : [s_0, t] \rightarrow S$  as follows:  $F(h) = f(\cdot + h) \in S$ ,  $h \in [s_0, t]$ , i.e.,  $F(h)(r) = f(r + h)$ ,  $r \in [0, t]$ .

If we prove that the mapping  $F$  is constant then we deduce (taking  $h = s_0$  and  $h = t$ ) that  $f(s_0 + \cdot) = f(t + \cdot) = f(t)$  in  $S$ . However, since  $f$  is continuous this implies that  $f$  is constant and finishes the proof.

The continuity of  $F$ , i.e., for any  $h \in [s_0, t]$ , we have

$$\lim_{h' \rightarrow h} \|F(h) - F(h')\|_S = \lim_{h' \rightarrow h} \int_0^t |f(r + h) - f(r + h')| dr = 0,$$

is clear, using the continuity of  $f$ . Let us prove that the left derivative of  $F$  is identically zero on  $(s_0, t]$ .

Using the flow property (iv) we find, for  $h, h' \in [s_0, t]$ ,  $h' < h$  and  $0 \leq r \leq t - h$ ,

$$\begin{aligned} & |f(r + h) - f(r + h')| \\ &= |\phi(r + h, t, g(r + h), \omega) - \phi(r + h, t, \phi(r + h', r + h, g(r + h'), \omega), \omega)|. \end{aligned} \quad (5.12)$$

Using (5.12) and changing variable we obtain (recall that  $f(r) = f(t)$ ,  $r \geq t$ )

$$\begin{aligned} & \int_0^t |f(r + h) - f(r + h')| dr \\ &= \int_0^{t-h} |\phi(r + h, t, g(r + h), \omega) - \phi(r + h, t, \phi(r + h', r + h, g(r + h'), \omega), \omega)| dr \\ & \quad + \int_{t-h}^{t-h'} |f(t) - f(r + h')| dr \\ &= \int_h^t |\phi(p, t, g(p), \omega) - \phi(p, t, \phi(p + h' - h, p, g(p + h' - h), \omega), \omega)| dp \\ & \quad + \int_{t-h}^{t-h'} |f(t) - f(r + h')| dr. \end{aligned} \quad (5.13)$$

In order to estimate  $\|F(h) - F(h')\|_S$  let us denote by  $\lambda_f$  the modulus of continuity of  $f$ . Since in the last integral  $t - h + h' \leq r + h' \leq t$  we have the estimate

$$\int_{t-h}^{t-h'} |f(t) - f(r + h')| dr \leq |h - h'| \lambda_f(|h - h'|)$$

and  $\lim_{r \rightarrow 0^+} \lambda_f(r) = 0$ . Taking into account that there exists a constant  $N_0 = N_0(x, T, \|b\|_0, \omega) \geq 1$  such that

$$|g(r)| + |\phi(r, u, g(r), \omega)| \leq N_0, \quad s_0 \leq r \leq u \leq T,$$

we find for  $p \in [h, t]$ ,  $n > 2d$  (see (5.4) and (5.9))

$$\begin{aligned} & |\phi(p, t, g(p), \omega) - \phi(p, t, \phi(p + h' - h, p, g(p + h' - h)), \omega), \omega)| \\ & \leq V_n(p, \omega) |g(p) - \phi(p + h' - h, p, g(p + h' - h)), \omega| \frac{n-2d}{n} N_0^{\frac{2d+1}{n}} \\ & \leq (2\|b\|_0)^{\beta(\frac{n-2d}{n})} [b]_{\beta, T}^{\frac{n-2d}{n}} V_n(p, \omega) |h' - h|^{(1+\beta)(\frac{n-2d}{n})} N_0^{\frac{2d+1}{n}}. \end{aligned}$$

Recall that  $V_n(p, \omega) \in [0, \infty]$  but  $\int_0^T V_n(p, \omega) dp < \infty$ . Using the previous inequality and (5.13) we obtain for  $h, h' \in [s_0, t]$ ,  $h' < h$

$$\begin{aligned} & \int_0^t |f(r + h) - f(r + h')| dr \\ & \leq C_0 |h' - h|^{(1+\beta)(\frac{n-2d}{n})} \int_0^T V_n(p, \omega) dp + |h - h'| \lambda_f(|h - h'|), \end{aligned} \quad (5.14)$$

where  $C_0 = C_0(\beta, \|b\|_{\beta, T}, \omega, T, x, n, d) > 0$ . Now we choose  $n$  large enough such that  $(1 + \beta)(\frac{n-2d}{n}) > 1$ . Dividing by  $|h - h'|$  and passing to the limit as  $h' \rightarrow h^-$  in (5.14) we find

$$\lim_{h' \rightarrow h^-} \frac{1}{|h - h'|} \|F(h) - F(h')\|_{L^1([0, t]; \mathbb{R}^d)} = 0.$$

This shows that there exists the left derivative of  $F$  in each  $h \in (s_0, t]$  and this derivative is identically zero on  $(s_0, t]$ . By the lemma mentioned at the beginning of III Step we obtain that  $F$  is constant. Thus  $f$  is constant on  $[s_0, t]$  and this finishes the proof.  $\blacksquare$

*Remark 5.2.* Note that if  $g : [s_0, \tau] \rightarrow \mathbb{R}^d$ ,  $\tau = \tau(\omega) \in (s_0, T]$ , solves (5.6) on  $[s_0, \tau]$  then we have  $g(\tau) = \phi(s_0, \tau, x, \omega)$ ,  $\omega \in \Omega'$ . Indeed applying (v) on  $[s_0, \tau]$  we can use that  $\int_{s_0}^\tau b(r, g(r)) dr = \int_{s_0}^\tau b(r, \phi(s_0, r, x, \omega)) dr$ .

*Remark 5.3.* It is a natural question if one can improve (5.4) in Theorem 5.1. A possible stronger assertion could be the following one: for each  $\alpha \in (0, 1)$  and  $N \in \mathbb{R}$  one can find  $C(\alpha, T, N, \omega) < \infty$  such that, for any  $x, y \in \mathbb{R}^d$ ,  $|x|, |y| < N$ ,

$$\sup_{s \in [0, T]} \sup_{t \in [s, T]} |\phi(s, t, x, \omega) - \phi(s, t, y, \omega)| \leq C(\alpha, T, N, \omega) |x - y|^\alpha, \quad \omega \in \Omega'. \quad (5.15)$$

This condition is stated as property 4 in Proposition 2.3 of [30] for SDEs (1.1) when  $L$  is a Wiener process and  $b \in L^q([0, T]; L^p(\mathbb{R}^d))$ ,  $d/p + 2/q < 1$ .

Assuming  $b \in L^\infty(0, T; C_b^{0, \beta}(\mathbb{R}^d; \mathbb{R}^d))$  we do not expect that (5.15) holds in general when  $L$  and  $b$  satisfy Hypotheses 1 and 2. Indeed a basic strategy to get (5.15) when  $L$  is a Wiener process is to use the Kolmogorov-Chentsov test to obtain a Hölder continuous dependence on  $(s, t, x)$ ; one cannot use this approach when  $L$  is a discontinuous process. Finally note that the proof of (5.15) given in [30] is not complete ((5.15) does not follow directly from estimate (4) in page 5 of [30] applying the Kolmogorov-Chentsov test).

Now we present two corollaries of Theorem 5.1 which deal with SDEs (1.1) with possibly unbounded  $b$ .

When  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is measurable and satisfies, for any  $\eta \in C_0^\infty(\mathbb{R}^d)$ ,  $b \cdot \eta \in L^\infty(0, T; C_b^{0, \beta}(\mathbb{R}^d; \mathbb{R}^d))$  we say that  $b \in L^\infty(0, T; C_{loc}^{0, \beta}(\mathbb{R}^d; \mathbb{R}^d))$ . By a localization procedure we get



**Corollary 5.4.** *Let  $b \in L^\infty(0, T; C_{loc}^{0, \beta}(\mathbb{R}^d; \mathbb{R}^d))$ ,  $\beta \in (0, 1]$ , and suppose that, for any  $\eta \in C_0^\infty(\mathbb{R}^d)$ , the Lévy process  $L$  and  $b \cdot \eta$  satisfy Hypotheses 1 and 2.*

*Then there exists an almost sure event  $\Omega''$  such that, for any  $\omega'' \in \Omega''$ ,  $x \in \mathbb{R}^d$ ,  $s_0 \in [0, T)$  and  $\tau = \tau(\omega'') \in (s_0, T]$ , if  $g_1, g_2 : [s_0, \tau) \rightarrow \mathbb{R}^d$  are càdlàg solutions of (5.6) when  $\omega = \omega''$ , starting from  $x$ , then  $g_1(r) = g_2(r)$ ,  $r \in [s_0, \tau)$ .*

*Proof.* Let  $\varphi \in C_0^\infty(\mathbb{R}^d)$  be such that  $\varphi = 1$  on  $\{|x| \leq 1\}$  and  $\varphi(x) = 0$  if  $|x| > 2$ . Set  $b_n(t, x) = b(t, x)\varphi(\frac{x}{n})$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  and  $n \geq 1$ . Consider for each  $n$  an almost sure event  $\Omega'_n$  related to  $b_n \in L^\infty(0, T; C_b^{0, \beta}(\mathbb{R}^d; \mathbb{R}^d))$  by Theorem 5.1; set  $\Omega'' = \cap_{n \geq 1} \Omega'_n$ . Suppose that  $g_1, g_2$  are solutions of (5.6) for a fixed  $\omega'' \in \Omega''$ . Let  $\tau_k^{(n)} = \tau_k^{(n)}(\omega'') = \inf\{t \in [s_0, \tau) : |g_k(t)| \geq n\}$ ,  $k = 1, 2$  (if  $|g_k(s)| < n$ , for any  $s \in [s_0, \tau)$  then we set  $\tau_k^{(n)} = \tau$ ). Define  $\tau^{(n)} = \tau_1^{(n)} \wedge \tau_2^{(n)}$  and note that on  $\Omega''$   $\tau^{(n)} \uparrow \tau$  as  $n \rightarrow \infty$ . Since on  $[s_0, \tau^{(n)}(\omega''))$  both  $g_1$  and  $g_2$  solve an equation like (5.6) with  $b$  replaced by  $b_n$  and  $\omega = \omega''$  we can apply (v) of Theorem 5.1 and conclude that  $g_1 = g_2$  on  $[s_0, \tau^{(n)}(\omega''))$ . Since this holds for any  $n \geq 1$  we get that  $g_1 = g_2$  on  $[s_0, \tau(\omega''))$ . ■

Next we construct  $\omega$  by  $\omega$  strong solutions to (1.1) when  $b$  is possibly unbounded. To simplify we deal with the initial time  $s = 0$ .

**Corollary 5.5.** *Suppose that  $L$  and  $b$  verify the assumptions of Corollary 5.4. Moreover assume that*

$$|b(t, x)| \leq C(1 + |x|), \quad x \in \mathbb{R}^d, \quad t \in [0, T], \quad (5.16)$$

*for some constant  $C > 0$ . Let  $x \in \mathbb{R}^d$  and  $s = 0$ . Then there exists a (unique) strong solution to (1.1) starting from  $x$ .*

*Proof.* We know that  $t \mapsto L_t(\omega)$  is càdlàg for any  $\omega \in \Omega'$ , where  $\Omega'$  is an almost sure event. When  $\omega \in \Omega'$  a standard argument based on the Ascoli-Arzelà theorem shows that there exists a continuous solution  $v = v(\cdot, \omega)$  to  $v(t) = x + \int_0^t b(s, v(s) + L_s(\omega))ds$  on  $[0, T]$ . We define  $v(t, \omega) = 0$ , if  $\omega \notin \Omega'$ ,  $t \in [0, T]$ . By using the function  $\varphi$  as in the proof of Corollary 5.4 we introduce  $b_n(t, x) = b(t, x)\varphi(\frac{x}{n})$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  and  $n \geq 1$ . According to Theorem 5.1 for each  $n$  there exists a function  $\phi_n$  as in (5.1) and an almost sure event  $\Omega'_n$  corresponding to  $b_n$  such that assertions (i)-(v) hold. Set  $\Omega'' = (\cap_{n \geq 1} \Omega'_n) \cap \Omega'$ .

Define  $g(t, \omega) = v(t, \omega) + L_t(\omega)$ ,  $t \in [0, T]$ ,  $\omega \in \Omega$ , and set  $\tau^{(n)} = \tau^{(n)}(\omega) = \inf\{t \in [0, T) : |g(t, \omega)| \geq n\}$  (if  $|g(s, \omega)| < n$ , for any  $s \in [0, T)$  then we set  $\tau^{(n)}(\omega) = T$ ). Note that on  $\Omega''$  we have  $\tau^{(n)} \uparrow T$  as  $n \rightarrow \infty$ .

Let  $\omega \in \Omega''$  and  $n \geq 1$ . Since on  $[0, \tau^{(n)}(\omega))$   $g(\cdot, \omega)$  solves an equation like (5.6) with  $s_0 = 0$  and  $b$  replaced by  $b_{n+k}$ ,  $k \geq 0$ , we can apply (v) of Theorem 5.1 and get that  $g(t, \omega) = \phi_{n+k}(0, t, x, \omega)$ , for any  $t \in [0, \tau^{(n)}(\omega))$ ,  $k \geq 0$ . Since  $\tau^{(n)} \uparrow T$  we deduce that, uniformly on compact sets of  $[0, T)$ , for any  $\omega \in \Omega''$ , we have  $\lim_{n \rightarrow \infty} \phi_n(0, t, x, \omega) = g(t, \omega)$ . It follows that  $g(t, \cdot)$  is  $\mathcal{F}_t^L$ -measurable, for any  $t \in [0, T)$ . By setting  $g(T, \omega) = x + \int_0^T b(r, g(r, \omega))dr + L_T(\omega)$ , we get that  $(g(t, \cdot))$  is a strong solution on  $[0, T]$ . ■

*Remark 5.6.* The previous condition (5.16) can be relaxed, by requiring that, for fixed  $x \in \mathbb{R}^d$ ,  $s = 0$  and  $\omega \in \Omega'$ , there exists a continuous solution to the integral equation  $v(t) = x + \int_0^t b(s, v(s) + L_s(\omega))ds$  on  $[0, T]$ . The assertion about existence and uniqueness of a strong solution starting from  $x$  remains true.

## 6 Uniqueness for SDEs driven by stable Lévy processes

In this section using also results from [24] and [25] we show that Theorem 5.1 can be applied to a class of SDEs driven by non-degenerate  $\alpha$ -stable type processes  $L$ . Let  $s \geq 0$ , we are considering

$$X_t(\omega) = x + \int_s^t b(X_u(\omega)) du + L_t(\omega) - L_s(\omega), \quad (6.1)$$

$x \in \mathbb{R}^d, d \geq 1, t \geq s$ , where  $b \in C_b^{0,\beta}(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\beta \in [0, 1]$ . We deal with *pure-jump Lévy process*  $L$  (without drift term), i.e., we assume that the generating triplet is  $(\nu, 0, 0)$  (i.e.,  $Q = 0$  and  $a = 0$  as in (2.5)). To state our assumptions on  $L$  we use the convolution semigroup  $(P_t)$  associated to  $L$  (or to its Lévy measure  $\nu$ ) and acting on  $C_b(\mathbb{R}^d)$ , i.e.,  $P_t : C_b(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d)$ ,  $t \geq 0$ ,

$$P_t f(x) = E[f(x + L_t)] = \int_{\mathbb{R}^d} f(x + z) \mu_t(dz), \quad t > 0, f \in C_b(\mathbb{R}^d), x \in \mathbb{R}^d,$$

where  $\mu_t$  is the law of  $L_t$ , and  $P_0 = I$  (cf. [28] or [1]). The generator  $\mathcal{L}$  of  $(P_t)$  is

$$\mathcal{L}g(x) = \int_{\mathbb{R}^d} (g(x + y) - g(x) - 1_{\{|y| \leq 1\}} \langle y, Dg(x) \rangle) \nu(dy), \quad x \in \mathbb{R}^d, \quad (6.2)$$

with  $g \in C_0^\infty(\mathbb{R}^d)$  (see Section 6.7 in [1] and Section 31 in [28]). We now consider the Blumenthal-Gettoor index  $\alpha_0 = \alpha_0(\nu)$  (see [5]):

$$\alpha_0 = \inf \left\{ \sigma > 0 : \int_{\{|x| \leq 1\}} |y|^\sigma \nu(dy) \right\} < \infty; \quad (6.3)$$

we always have  $\alpha_0 \in [0, 2]$ . In the sequel we require that  $\alpha_0 \in (0, 2)$ . Similarly to [25] we make the following assumption on the Lévy measure  $\nu$ .

**Hypothesis 3.** Let  $\alpha_0 \in (0, 2)$ . The convolution semigroup  $(P_t)$  verifies:  $P_t(C_b(\mathbb{R}^d)) \subset C_b^1(\mathbb{R}^d)$ ,  $t > 0$ , and, moreover, there exists  $c_{\alpha_0} = c_{\alpha_0}(\nu) > 0$  such that

$$\sup_{x \in \mathbb{R}^d} |DP_t f(x)| \leq c_{\alpha_0} t^{-\frac{1}{\alpha_0}} \cdot \sup_{x \in \mathbb{R}^d} |f(x)|, \quad t \in (0, 1], f \in C_b(\mathbb{R}^d). \quad \blacksquare \quad (6.4)$$

Note that Hypothesis 3 implies both Hypotheses 1 and 2 in [25] (taking  $\alpha = \alpha_0$ ). Indeed since  $\alpha_0 \in (0, 2)$  we have  $\int_{\{|x| \leq 1\}} |y|^\sigma \nu(dy) < \infty$ , for  $\sigma > \alpha_0$ . To check the validity of the gradient estimate (6.4) we only mention a criterion which is given in [25]; it is based on Theorem 1.3 in [29].

**Theorem 6.1.** *Let  $L$  be a pure-jump Lévy process. A sufficient condition in order that (6.4) holds when  $\alpha_0$  replaced by  $\gamma \in (0, 2)$  is the following one: the Lévy measure  $\nu$  of  $L$  verifies:  $\nu(B) \geq \nu_1(B)$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$ , where  $\nu_1$  is a Lévy measure on  $\mathbb{R}^d$  such that its corresponding symbol  $\psi_1(h) = -\int_{\mathbb{R}^d} (e^{i\langle h, y \rangle} - 1 - i\langle h, y \rangle 1_{\{|y| \leq 1\}}(y)) \nu_1(dy)$ , satisfies, for some positive constants  $c_1, c_2$  and  $M$ ,*

$$c_1 |x|^\gamma \leq \operatorname{Re} \psi_1(x) \leq c_2 |x|^\gamma, \quad \text{when } |x| > M. \quad (6.5)$$

*Examples 6.2.* The next examples of  $\alpha$ -stable type Lévy processes are also considered in [25]. It is easy to check that in each example  $\alpha_0 = \alpha \in (0, 2)$ . Thanks to Theorem 6.1 also (6.4) holds in each example.

Consider the following Lévy measure  $\tilde{\nu}$ :

$$\tilde{\nu}(B) = \int_0^r \frac{dt}{t^{1+\alpha}} \int_S 1_B(t\xi) \mu(d\xi), \quad B \in \mathcal{B}(\mathbb{R}^d) \quad (6.6)$$

(cf. Example 1.5 of [29] with the index  $\beta$  of [29] which is equal to  $\infty$ ). Here  $r > 0$  is fixed;  $\mu$  is a non-degenerate finite non-negative measure on  $\mathcal{B}(\mathbb{R}^d)$  with support on the unit sphere  $S$  (non-degeneracy of  $\mu$  is equivalent to say that its support is not contained in a proper linear subspace of  $\mathbb{R}^d$ ),  $\alpha \in (0, 2)$ . The Lévy measure  $\tilde{\nu}$  verifies Hypothesis 3 since its symbol  $\tilde{\psi}$  verifies (6.5) with  $\gamma = \alpha$ . This was already remarked in page 1146 of [29]. We only note that, if  $h \neq 0$ , we have

$$\operatorname{Re} \tilde{\psi}(h) = \int_0^r \frac{dt}{t^{1+\alpha}} \int_S \left[ 1 - \cos \left( \left\langle \frac{h}{|h|}, t|h|\xi \right\rangle \right) \right] \mu(d\xi).$$

By changing variable  $s = t|h|$  after some computations one arrives at (6.5).

Moreover Hypothesis 2 holds. Note that  $\int_{\{|x|>1\}} |y|^\theta \tilde{\nu}(dy) < \infty$ ,  $\theta \in (0, \alpha)$ . Using also  $\tilde{\nu}$  we find that the next examples of Lévy processes verify Hypotheses 2 and 3.

(i) *L is a non-degenerate symmetric  $\alpha$ -stable process* (see, for instance, [28] and the references therein). In this case  $\nu(B) = \int_0^\infty \frac{dt}{t^{1+\alpha}} \int_S 1_B(t\xi) \mu(d\xi)$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$ ,  $\alpha \in (0, 2)$ , where  $\mu$  is as in (6.6). A standard rotationally invariant  $\alpha$ -stable process  $L$  belongs to this class since its Lévy measure has density  $\frac{c}{|x|^{d+\alpha}}$  (with respect to the Lebesgue measure in  $\mathbb{R}^d$ ).

(ii) *L is a  $\alpha$ -stable tempered process of special form.* Here

$$\nu(B) = \int_0^\infty \frac{e^{-t} dt}{t^{1+\alpha}} \int_S 1_B(t\xi) \mu(d\xi), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where  $\mu$  is as in (6.6),  $\alpha \in (0, 2)$ .

Note that in (i) and (ii) we have  $\nu(B) \geq e^{-1} \tilde{\nu}(B)$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$ , where  $\tilde{\nu}$  is given in (6.6) with  $r = 1$ .

(iii) *L is a truncated  $\alpha$ -stable process.* In this case  $\nu(B) = c \int_{\{|x| \leq 1\}} \frac{1_B(x)}{|x|^{d+\alpha}} dx$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$ ,  $\alpha \in (0, 2)$ .

(iv) *L is a relativistic  $\alpha$ -stable process* (cf. [27] and see the references therein). Here  $\psi(h) = (|h|^2 + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} - m$ , for some  $m > 0$ ,  $\alpha \in (0, 2)$ ,  $h \in \mathbb{R}^d$ , and so (6.4) holds. Moreover by Lemma 2 in [27] we know that  $\nu$  has the density  $C_{\alpha,d} |x|^{-d-\alpha} e^{-m^{1/\alpha} |x|} \cdot \phi(m^{1/\alpha} |x|)$ ,  $x \neq 0$ , with  $0 \leq \phi(s) \leq c_{\alpha,d,m} (s^{\frac{d-1+\alpha}{2}} + 1)$ ,  $s \geq 0$ . Hence  $\alpha = \alpha_0$  and also Hypothesis 2 holds for any  $\theta > 0$ .

## 6.1 Results on strong existence and uniqueness by using solutions of related Kolmogorov equations

We first present results on strong existence and uniqueness for (6.1) when  $s = 0$  which are special cases of Lemma 5.2 and Theorem 5.3 in [25]. Then we study  $L^p$ -dependence from the initial condition  $x$  following Theorem 4.3 in [24]. Finally in Theorem 6.6 we will consider the general case when  $s \in [0, T]$ .

All these theorems do not require the gradient estimates (6.4). However they assume the Blumenthal-Gettoor index  $\alpha_0 \in (0, 2)$ ,  $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$  and classical solvability of the following Kolmogorov type equation:

$$\lambda u(x) - \mathcal{L}u(x) - Du(x)b(x) = b(x), \quad x \in \mathbb{R}^d, \quad (6.7)$$

where  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is given in (6.1),  $\mathcal{L}$  in (6.2) and  $\lambda > 0$ ; the equation is intended componentwise, i.e.,  $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and, setting  $\mathcal{L}_b = \mathcal{L} + b(x) \cdot D$ ,

$$\lambda u_k(x) - \mathcal{L}_b u_k(x) = b_k(x), \quad k = 1, \dots, d, \quad (6.8)$$

with  $u(x) = (u_k(x))_{k=1, \dots, d}$  and  $b(x) = (b_k(x))_{k=1, \dots, d}$ . The approach to get strong uniqueness passing through solutions to (6.7) is similar to the one used in Section 2 of [12] (see also [35]).

Remark that  $\mathcal{L}g(x)$  in (6.2) is well defined even for  $g \in C_b^{1+\gamma}(\mathbb{R}^d)$  if  $\alpha_0 < 1 + \gamma$  and  $\gamma \in [0, 1]$  (cf. formula (13) in [25]). Indeed when  $|y| \leq 1$  we can use the bound  $|g(y+x) - g(x) - y \cdot Dg(x)| \leq [Dg]_\gamma |y|^{1+\gamma}$ ,  $x \in \mathbb{R}^d$ .

In addition  $\mathcal{L}g \in C_b(\mathbb{R}^d)$  when  $g \in C_b^{1+\gamma}(\mathbb{R}^d)$  and  $1 + \gamma > \alpha_0$ . The next result is stated in Theorem 5.3 of [25] in a more general form which also shows the differentiability of solutions with respect to  $x$  and the homeomorphism property.

**Theorem 6.3.** *Let  $L$  be any Lévy process on  $(\Omega, \mathcal{F}, P)$  with generating triplet  $(\nu, 0, 0)$  such that  $\alpha_0 = \alpha_0(\nu) \in (0, 2)$  (see (6.3)) and let  $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$  in (6.1). Suppose that, for some  $\lambda > 0$ , there exists  $u = u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\gamma \in (0, 1)$  and  $2\gamma > \alpha_0$ , which solves (6.7). Moreover, assume  $\|Du_\lambda\|_0 < 1/3$ .*

*Then on  $(\Omega, \mathcal{F}, P)$ , for any  $x \in \mathbb{R}^d$ , there exists a pathwise unique strong solution  $(X_t^x)_{t \geq 0}$  to (6.1) when  $s = 0$ .*

Next we formulate a special case of Lemma 5.2 in [25]. It uses the stochastic integral against the compensated Poisson random measure  $\tilde{N}$  (see, for instance, [20]).

**Lemma 6.4.** *Under the same hypotheses of Theorem 6.3 let  $T > 0$  and suppose that  $(X_t^x)_{t \in [0, T]}$  is a strong solution of (6.1) on  $[0, T]$  when  $s = 0$  (starting from  $x \in \mathbb{R}^d$ ), then, using  $u_\lambda$  of Theorem 6.3, we have,  $P$ -a.s., for any  $t \in [0, T]$ ,*

$$\begin{aligned} & u_\lambda(X_t^x) - u_\lambda(x) \\ &= x + L_t - X_t^x + \lambda \int_0^t u_\lambda(X_s^x) ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u_\lambda(X_{s-}^x + y) - u_\lambda(X_{s-}^x)] \tilde{N}(ds, dy). \end{aligned} \quad (6.9)$$

*Proof.* The assertion is stated in Lemma 5.2 of [25] for weak solutions  $(X_t^x)_{t \geq 0}$  with the condition  $1 + \gamma > \alpha_0$ ,  $\gamma \in (0, 1]$ . Clearly such lemma works also for strong solutions  $(X_t^x)_{t \in [0, T]}$  which solves (6.1) on  $[0, T]$  (the proof is based on Itô's formula for  $u_\lambda(X_t^x)$ ); further the condition  $2\gamma > \alpha_0$  of Theorem 6.3 implies  $1 + \gamma > \alpha_0$ . ■

To prove Davie's uniqueness for (6.1) we need the following  $L^p$ -continuity of the solutions w.r.t. initial conditions.

**Theorem 6.5.** *Under the same hypotheses of Theorem 6.3 let  $T > 0$ ,  $s = 0$ , and consider two strong solutions  $(X_t^x)_{t \in [0, T]}$  and  $(X_t^y)_{t \in [0, T]}$  of (6.1) on  $[0, T]$  which are*

defined on  $(\Omega, \mathcal{F}, P)$ , starting from  $x$  and  $y \in \mathbb{R}^d$  respectively. For any  $t \in [0, T]$ ,  $p \geq 2$ , we have

$$E[\sup_{0 \leq s \leq t} |X_s^x - X_s^y|^p] \leq C(t) |x - y|^p, \quad (6.10)$$

with  $C(t) = C(t, \nu, p, \lambda, d, \gamma, \|u_\lambda\|_{C_b^{1+\gamma}}) > 0$  which is independent of  $x$  and  $y$ ; here  $u_\lambda$  is as in Theorem 6.3 (further  $C(t, \nu, p, \lambda, d, \gamma, \cdot)$  is increasing).

*Proof.* The proof follows the one of (i) in Theorem 4.3 of [24]. We only give a sketch of the proof here. We set  $X = X^x$ ,  $Y = X^y$  and  $u = u_\lambda$ . We have from Lemma 6.4,  $P$ -a.s., using that  $\|Du\|_0 \leq 1/3$ ,  $|X_t - Y_t| \leq \frac{3}{2}(\Gamma_1(t) + \Gamma_2(t) + \Gamma_3(t) + \Gamma_4)$ , where

$$\Gamma_1(t) = \left| \int_0^t \int_{\{|z|>1\}} [u(X_{s-} + z) - u(X_{s-}) - u(Y_{s-} + z) + u(Y_{s-})] \tilde{N}(ds, dz) \right|,$$

$$\Gamma_2(t) = \lambda \int_0^t |u(X_s) - u(Y_s)| ds,$$

$$\Gamma_3(t) = \left| \int_0^t \int_{\{|z|\leq 1\}} [u(X_{s-} + z) - u(X_{s-}) - u(Y_{s-} + z) + u(Y_{s-})] \tilde{N}(ds, dz) \right|,$$

$$\Gamma_4 = |u(x) - u(y)| + |x - y| \leq \frac{4}{3} |x - y|. \text{ Remark that, } P\text{-a.s.,}$$

$$\sup_{0 \leq r \leq t} |X_r - Y_r|^p \leq C_1 |x - y|^p + C_1 \sum_{j=1}^3 \sup_{0 \leq r \leq t} \Gamma_j(r)^p.$$

By the Hölder inequality,  $\sup_{0 \leq r \leq t} \Gamma_2(r)^p \leq C_2 t^{p-1} \int_0^t \sup_{0 \leq s \leq r} |X_s - Y_s|^p dr$ , where  $C_2 = C_2(p, \lambda, \|u_\lambda\|_{C_b^{1+\gamma}})$ . To estimate  $\Gamma_1$  and  $\Gamma_3$  we use  $L^p$ -estimates for stochastic integrals against  $\tilde{N}$  (cf. [20, Theorem 2.11] or the proof of Proposition 6.6.2 in [1]).

We find, since  $|u(X_{s-} + z) - u(Y_{s-} + z) + u(Y_{s-}) - u(X_{s-})| \leq \frac{2}{3} |X_{s-} - Y_{s-}|$ , setting  $A = \{|z| > 1\}$ ,

$$\begin{aligned} & E\left[ \sup_{0 \leq r \leq t} \Gamma_1(r)^p \right] \\ & \leq C_3 E\left[ \left( \int_0^t ds \int_A |u(X_{s-} + z) - u(Y_{s-} + z) + u(Y_{s-}) - u(X_{s-})|^2 \nu(dy) \right)^{p/2} \right] \\ & \quad + C_3 E \int_0^t ds \int_A |u(X_{s-} + z) - u(Y_{s-} + z) + u(Y_{s-}) - u(X_{s-})|^p \nu(dy) \\ & \leq C_4 (1 + t^{p/2-1}) \int_0^t E\left[ \sup_{0 \leq r \leq s} |X_r - Y_r|^p \right] ds, \end{aligned}$$

where  $C_3 = \int_{\{|z|>1\}} \nu(dz) + \left( \int_{\{|z|>1\}} \nu(dz) \right)^{p/2}$ . To treat  $\Gamma_3$  we need the hypothesis  $2\gamma > \alpha_0$ . By  $L^p$ -estimates of stochastic integrals and using Lemma 4.1 in [24] we get

$$\begin{aligned} E\left[ \sup_{0 \leq r \leq t} \Gamma_3(r)^p \right] & \leq C_5 \|u\|_{C_b^{1+\gamma}}^p E\left[ \left( \int_0^t dr \int_{\{|z|\leq 1\}} |X_r - Y_r|^2 |z|^{2\gamma} \nu(dz) \right)^{p/2} \right] \\ & \quad + C_5 \|u\|_{C_b^{1+\gamma}}^p E \int_0^t |X_r - Y_r|^p dr \int_{\{|z|\leq 1\}} |z|^{\gamma p} \nu(dz). \end{aligned}$$

Note that  $\int_{\{|z|\leq 1\}} |z|^{\gamma p} \nu(dz) < \infty$ , since  $p \geq 2$  and  $2\gamma > \alpha_0$ . Collecting the previous estimates, we arrive at

$$E\left[ \sup_{0 \leq r \leq t} |X_r - Y_r|^p \right] \leq C_6 |x - y|^p + C_6 (1 + t^{p-1}) \int_0^t E\left[ \sup_{0 \leq r \leq s} |X_r - Y_r|^p \right] ds,$$

$C_6 = C_6(\nu, p, \lambda, d, \gamma) > 0$ . By the Gronwall lemma we obtain the assertion with  $C(t) = C_6 \exp(C_6(1 + t^{p-1}))$ .  $\blacksquare$

As a consequence of the previous results we get

**Theorem 6.6.** *Under the same hypotheses of Theorem 6.3 let  $T > 0$  and  $s \in [0, T]$ . Then, for any  $x \in \mathbb{R}^d$ , there exists a pathwise unique strong solution  $\tilde{X}^{s,x} = (\tilde{X}_t^{s,x})_{t \in [0, T]}$  to (6.1) on  $(\Omega, \mathcal{F}, P)$  (recall that  $\tilde{X}_t^{s,x} = x$  for  $t \leq s$ ). Moreover if  $U^{s,x}$  and  $U^{s,y}$  are two strong solutions on  $[0, T]$  defined on  $(\Omega, \mathcal{F}, P)$  and starting at  $x$  and  $y$ , then we have, for  $p \geq 2$ ,*

$$\sup_{s \in [0, T]} E[ \sup_{s \leq t \leq T} |U_t^{s,x} - U_t^{s,y}|^p ] \leq C(T) |x - y|^p, \quad x, y \in \mathbb{R}^d, \quad (6.11)$$

where  $C(T) = C(T, \nu, p, \lambda, d, \gamma, \|u_\lambda\|_{C_b^{1+\gamma}}) > 0$  as in (6.10).

*Proof. Existence.* Let us fix  $s \in [0, T]$  and consider the new process  $L^{(s)} = (L_t^{(s)})$  on  $(\Omega, \mathcal{F}, P)$ ,  $L_t^{(s)} = L_{s+t} - L_s$ ,  $t \geq 0$ . This is a Lévy process with the same generating triplet of  $L$  and is independent of  $\mathcal{F}_s^L$  (see Proposition 10.7 in [28]). According to Theorem 6.3 there exists a unique strong solution to

$$X_t = x + \int_0^t b(X_r) dr + L_t^{(s)}, \quad t \geq 0, \quad (6.12)$$

which we denote by  $(X_{t, L^{(s)}}^x)$  to stress its dependence on  $L^{(s)}$ . Note that, for any  $t \geq 0$ ,  $X_{t, L^{(s)}}^x$  is measurable with respect to  $\mathcal{F}_t^{L^{(s)}} = \mathcal{F}_{s, t+s}^L$ . Let us define a new process with càdlàg paths  $(\tilde{X}_t^{s,x})_{t \in [0, T]}$ ,

$$\tilde{X}_t^{s,x} = X_{t-s, L^{(s)}}^x, \quad \text{for } s \leq t \leq T; \quad \tilde{X}_t^{s,x} = x, \quad 0 \leq t \leq s. \quad (6.13)$$

Writing  $V_t = \tilde{X}_t^{s,x}$ ,  $t \in [0, T]$ , to simplify notation, we note that  $V_t$  is  $\mathcal{F}_{s, t}^L$ -measurable,  $t \geq s$ . Moreover it solves equation (6.1); indeed, for  $t \in [s, T]$ ,

$$V_t = X_{t-s, L^{(s)}}^x = x + \int_0^{t-s} b(X_{r, L^{(s)}}^x) dr + L_t - L_s = x + \int_s^t b(V_r) dr + L_t - L_s.$$

Uniqueness. Let  $(U_t^{s,x})$  be another strong solution. We have,  $P$ -a.s., for  $s \leq t \leq T$ ,

$$\begin{aligned} U_{t-s+s}^{s,x} &= x + \int_s^t b(U_r^{s,x}) dr + L_t - L_s \\ &= x + \int_0^{t-s} b(U_{r+s}^{s,x}) dr + L_t - L_s = x + \int_0^{t-s} b(U_{r+s}^{s,x}) dr + L_{t-s}^{(s)}. \end{aligned}$$

Hence  $(U_{r+s}^{s,x})_{r \in [0, T-s]}$  solves (6.12) on  $[0, T-s]$ . By (6.10) we get

$$P(U_{r+s}^{s,x} = X_{r, L^{(s)}}^x, r \in [0, T-s]) = P(U_{r+s}^{s,x} = \tilde{X}_{r+s}^{s,x}, r \in [0, T-s]) = 1.$$

This shows the assertion.

$L^p$ -estimates. We have for any fixed  $s \in [0, T]$ ,  $p \geq 2$ ,  $E[\sup_{s \leq t \leq T} |U_t^{s,x} - U_t^{s,y}|^p] = E[\sup_{s \leq t \leq T} |X_{t-s, L(s)}^x - X_{t-s, L(s)}^y|^p]$  by uniqueness. Using (6.10) we get

$$\begin{aligned} \sup_{s \in [0, T]} E[\sup_{s \leq t \leq T} |U_t^{s,x} - U_t^{s,y}|^p] &= \sup_{s \in [0, T]} E[\sup_{s \leq t \leq T} |X_{t-s, L(s)}^x - X_{t-s, L(s)}^y|^p] \\ &\leq \sup_{s \in [0, T]} E[\sup_{t \in [0, T]} |X_{t, L(s)}^x - X_{t, L(s)}^y|^p] \leq C(T) |x - y|^p. \end{aligned}$$

■

## 6.2 A Davie's type uniqueness result when $\alpha_0 \in [1, 2)$

Here we prove a Davie's type uniqueness result for (6.1) (cf. Theorem 5.1). We consider the Blumenthal-Gettoor index  $\alpha_0 \in [1, 2)$  (see (6.3)) and assume as in [24] and [25] that  $b \in C_b^{0, \beta}(\mathbb{R}^d, \mathbb{R}^d)$  with  $\beta \in (1 - \frac{\alpha_0}{2}, 1]$ .

To check Hypothesis 1 we will use Theorem 6.6 and the following purely analytic result (see Theorem 4.3 in [25]; its the proof follows the one in Theorem 3.4 of [24]). Note that the next hypothesis  $\alpha_0 + \beta < 2$  could be dropped. Moreover, to simplify we have only considered the case  $\lambda \geq 1$  instead of  $\lambda > 0$ .

**Theorem 6.7.** *Assume Hypothesis 3 with  $\alpha_0 = \alpha_0(\nu) \geq 1$ . Let  $0 < \beta < 1$  with  $\alpha_0 + \beta \in (1, 2)$  and consider  $\mathcal{L}$  in (6.2). Then, for any  $\lambda \geq 1$ ,  $f \in C_b^\beta(\mathbb{R}^d)$ , there exists a unique solution  $w_\lambda \in C_b^{\alpha_0 + \beta}(\mathbb{R}^d)$  to*

$$\lambda w(x) - \mathcal{L}w(x) - b(x) \cdot Dw(x) = f(x), \quad x \in \mathbb{R}^d \quad (6.14)$$

Moreover, there exists  $C_0 = C_0(\alpha_0(\nu), d, \beta, \|b\|_{C_b^\beta}, \nu) > 0$  such that

$$\lambda \|w_\lambda\|_0 + [Dw_\lambda]_{C_b^{\alpha_0 + \beta - 1}} \leq C_0 \|f\|_{C_b^\beta}, \quad \lambda \geq 1. \quad (6.15)$$

Finally, we have  $\|Dw_\lambda\|_0 < 1/3$ , for any  $\lambda \geq \lambda_0(d, \|b\|_{C_b^\beta}, \alpha_0(\nu), \beta, \nu) \geq 1$ .

*Proof.* We only make some comments on  $C_0$  and  $\lambda_0$ . Let us first consider  $C_0$ . To see that  $C_0 = C_0(\alpha_0(\nu), d, \beta, \|b\|_{C_b^\beta}, \nu)$  we look into the proof of Theorem 4.3 in [25]. In such proof the Schauder estimates (6.15) are first established as apriori estimates by a localization procedure. This method is based on Schauder estimates already proved in the constant coefficients case, i.e., when  $b(x) = k$ ,  $x \in \mathbb{R}^d$  (see Theorem 4.2 in [25]). The Schauder constant  $C_0$  depends on the Schauder constant  $c$  appearing in formula (16) of Theorem 4.2 in [25] when  $\lambda \geq 1$ . Such constant  $c$  depends on  $\alpha_0(\nu), \beta, d$  and also on the constant  $c_{\alpha_0}$  of the gradient estimates (6.4) (see, in particular, estimates (18)-(21) in the proof of Theorem 4.2 in [25]).

Let us consider  $\lambda_0$ . Recall the simple estimate  $\|Dw_\lambda\|_0 \leq N [Dw_\lambda]_{C_b^{\alpha_0 + \beta - 1}}^{\frac{1}{\alpha_0 + \beta}} \|w_\lambda\|_0^{\frac{\alpha_0 + \beta - 1}{\alpha_0 + \beta}}$ , where  $N = N(\alpha_0, \beta, d)$  (cf. the proof of Theorem 3.4 in [24]). By (6.15) we get  $\|Dw_\lambda\|_0 \leq NC_0 \lambda^{-\frac{\alpha_0 + \beta - 1}{\alpha_0 + \beta}} \|f\|_{C_b^\beta}$ ,  $\lambda \geq 1$ , and the assertion follows by choosing  $\lambda_0 > 1 \vee (3NC_0)^{\frac{\alpha_0 + \beta}{\alpha_0 + \beta - 1}}$ . ■

Currently we do not know if the statements in Theorem 6.7 hold also when  $\alpha_0 \in (0, 1)$  (maintaining all the other assumptions).

Now we apply Theorem 5.1 to get Davie's type uniqueness for the SDE (6.1).

**Theorem 6.8.** *Let  $L$  be a  $d$ -dimensional Lévy process on  $(\Omega, \mathcal{F}, P)$  with generating triple  $(\nu, 0, 0)$  satisfying Hypothesis 3 with  $\alpha_0 \in [1, 2)$ . Suppose also that  $\int_{\{|x|>1\}} |y|^\theta \nu(dy) < \infty$ , for some  $\theta > 0$ . Let us consider (6.1) with  $b \in C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d)$  and  $\beta \in (1 - \frac{\alpha_0}{2}, 1]$ .*

*Then  $L$  satisfies Hypothesis 1 and, for any  $T > 0$ , there exists a function  $\phi$  as in Theorem 5.1 such that assertions (i)-(v) hold on some almost sure event  $\Omega'$ .*

*Proof.* When  $\beta = 1$  Hypothesis 1 is clearly satisfied. Let us consider  $\beta \in (1 - \frac{\alpha_0}{2}, 1)$ . Since  $C_b^{\beta'}(\mathbb{R}^d, \mathbb{R}^d) \subset C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$  when  $0 < \beta \leq \beta' \leq 1$ , we may assume that  $1 - \frac{\alpha_0}{2} < \beta < 2 - \alpha_0$ . To verify Hypothesis 1 we use Theorems 6.7 and 6.6. By Theorem 6.7 we have a solution  $u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$  to (6.7) with  $\gamma = \alpha_0 - 1 + \beta \in (0, 1)$  for any  $\lambda \geq 1$ . Note that  $2\gamma = 2\alpha_0 - 2 + 2\beta > \alpha_0$ . Choosing  $\lambda = \lambda_0(d, \|b\|_{C_b^\beta}, \alpha_0(\nu), \beta)$  we obtain that also  $\|Du_\lambda\| < 1/3$  holds.

Using Theorem 6.6 we can check the validity of (2.6). Note that the constant  $C(T)$  appearing in (6.11) depends on  $T, \nu, p, \alpha_0(\nu), \lambda, d, \gamma$  and  $\|u_\lambda\|_{C_b^{1+\gamma}}$ . However by Theorem 6.7  $\gamma = \alpha_0 - 1 + \beta$ ,  $\lambda = \lambda_0(d, \|b\|_{C_b^\beta}, \alpha_0, \beta)$  and  $\|u_\lambda\|_{C_b^{1+\gamma}} = \|u_\lambda\|_{C_b^{\alpha_0+\beta}} \leq N(\alpha_0, \beta, d) C_0 \|b\|_{C_b^\beta}$  where  $C_0$  appears in the Schauder estimates (6.15). It follows that  $C(T)$  in (6.11) has the right dependence on  $d, p, \beta, \nu, \|b\|_{C_b^\beta}$  and  $T$  as required in (2.6). To finish the proof we apply Theorem 5.1 since Hypotheses 1 and 2 hold.  $\blacksquare$

*Remark 6.9.* Theorem 6.8 shows that under suitable assumptions on  $L$  and  $b$  Davie's uniqueness (or path-by-path uniqueness) holds for the SDE (1.1). Moreover, the unique strong solution is given by a function  $\phi$  which satisfies all the assertions of Theorem 5.1, including (5.2) and (5.4), for any  $\omega \in \Omega'$ , where  $\Omega'$  is an almost sure event independent of  $s, t$  and  $x$ . There are no similar results in the literature on stochastic flows for SDEs (1.1) driven by stable type processes (cf. [24], [25] and the recent paper [6] which contains the most general available results about existence and  $C^1$ -regularity of stochastic flow).

### 6.3 Davie's type uniqueness when $\alpha_0 = \alpha \in (0, 1)$

Here we only consider the SDE (6.1) when  $L = L_\alpha$  is a symmetric rotationally invariant  $\alpha$ -stable process with  $\alpha \in (0, 1)$  (the case of  $\alpha \in [1, 2)$  is already treated in Theorem 6.8). For each  $\alpha \in (0, 1)$  its Lévy measure  $\nu = \nu_\alpha$  has density  $\frac{c_{\alpha,d}}{|y|^{d+\alpha}}$ ,  $y \neq 0$ , and its generator  $\mathcal{L} = \mathcal{L}^{(\alpha)}$  (see (6.2)) coincides with the fractional Laplacian  $-(-\Delta)^{\alpha/2}$  (see Example 32.7 in [28]). Note that, for any  $g \in C_b^1(\mathbb{R}^d)$ , the mapping:

$$x \mapsto \mathcal{L}g(x) = c_{\alpha,d} \int_{\mathbb{R}^d} \frac{g(x+y) - g(x)}{|y|^{d+\alpha}} dy \text{ belongs to } C_b(\mathbb{R}^d). \quad (6.16)$$

Clearly  $\alpha = \alpha_0$  (see (3.1)). Using Theorem 6.6 of the previous section together with Theorem 6.11 we can apply Theorem 5.1 and obtain

**Theorem 6.10.** *Let  $L$  be a  $d$ -dimensional symmetric rotationally invariant  $\alpha$ -stable process with  $\alpha \in (0, 1)$  defined on  $(\Omega, \mathcal{F}, P)$ . Let us consider the SDE (6.1) with  $b \in C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d)$  and  $\beta \in (1 - \frac{\alpha}{2}, 1]$ .*

*Then  $L$  satisfies Hypotheses 1 and 2 and, for any  $T > 0$ , there exists a function  $\phi$  as in Theorem 5.1 such that assertions (i)-(v) hold on some almost sure event  $\Omega'$ .*



We first state a result which is related to Theorem 6.7. It shows sharp  $C_b^{\alpha+\beta}$ -regularity of solutions to (6.14). The proof is based on Theorem 1.1 in [31].

**Theorem 6.11.** *Let us consider the fractional Laplacian  $\mathcal{L}$  given in (6.16) with  $\alpha \in (0, 1)$ . Let  $\beta \in (0, 1)$  such that  $\alpha + \beta > 1$ . Then, for any  $\lambda \geq 1$ ,  $f \in C_b^\beta(\mathbb{R}^d)$ , there exists a unique solution  $w = w_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$  to (6.14). Moreover, there exists  $C_0 = C_0(\alpha, d, \beta, \|b\|_{C_b^\beta}) > 0$  such that*

$$\lambda \|w_\lambda\|_0 + [Dw_\lambda]_{C_b^{\alpha+\beta-1}} \leq C_0 \|f\|_{C_b^\beta}, \quad \lambda \geq 1. \quad (6.17)$$

Finally, we have  $\|Dw_\lambda\|_0 < 1/3$ , for any  $\lambda \geq \lambda_0$ , with  $\lambda_0(d, \|b\|_{C_b^\beta}, \alpha, \beta) \geq 1$ .

*Proof.* The uniqueness follows by the maximum principle (see Proposition 3.2 in [24] or Proposition 4.1 in [25]) which states that  $\lambda \|w_\lambda\|_0 \leq \|f\|_0$ . Let  $\mathcal{L}_b$  be the fractional Laplacian  $\mathcal{L}$  plus the drift  $b$  (i.e.,  $\mathcal{L}_b = \mathcal{L} + b \cdot D$ ). The proof proceeds in some steps.

*I step.* Let  $\lambda \geq 1$ . We provide apriori estimates for classical  $C_b^1$ -solutions  $u$  to  $\lambda u - \mathcal{L}_b u = f$  on  $\mathbb{R}^d$  (with  $f \in C_b^\beta(\mathbb{R}^d)$ ,  $b \in C_b^\beta(\mathbb{R}^d; \mathbb{R}^d)$  and  $\alpha + \beta > 1$ ).

Let  $u = u_\lambda \in C_b^1(\mathbb{R}^d)$  be a solution to  $\lambda u - \mathcal{L}_b u = f$  on  $\mathbb{R}^d$ ; in the sequel we will consider open balls  $B_r(x_0)$  of center  $x_0 \in \mathbb{R}^d$  and radius  $r > 0$ . Let  $x_0 \in \mathbb{R}^d$ . One can define  $v(x) = u(x + x_0)$ ,  $x \in \mathbb{R}^d$ . Since  $\mathcal{L}v(x) = \mathcal{L}u(x + x_0)$ ,  $x \in \mathbb{R}^d$ , we get that  $v \in C_b^1(\mathbb{R}^d)$  solves  $\lambda v - \mathcal{L}_{b_0} v = f_0$  on  $\mathbb{R}^d$  where  $\mathcal{L}_{b_0}$  has the drift  $b_0(\cdot) = b(\cdot + x_0)$  and  $f_0(\cdot) = f(\cdot + x_0)$ .

Setting  $\tilde{v}(t, x) = e^{\lambda t} v(x)$ ,  $\tilde{f}_0(t, x) = e^{\lambda t} f_0(x)$ ,  $t \in [-1, 0]$ ,  $x \in \mathbb{R}^d$ , we see that  $\tilde{v}$  is a bounded solution of

$$\partial_t \tilde{v} - \mathcal{L}_{b_0} \tilde{v} = \tilde{f}_0 \quad \text{on } [-1, 0] \times B_1(0)$$

according to the definition of viscosity solution given at the beginning of Section 3.1 in [31]. Hence we can apply Theorem 1.1 in [31] to  $\tilde{v}$ . Recall that in the Silvestre notations his  $s \in (0, 1)$  is our  $\alpha/2$  and his  $\alpha \in (0, 2s)$  corresponds with our  $\alpha + \beta - 1$ . We deduce by [31] that  $\tilde{v}(t, \cdot) \in C^{\alpha+\beta}(B_{1/2}(0))$  and moreover

$$\begin{aligned} \|v\|_{C^{\alpha+\beta}(B_{1/2}(0))} &= \|\tilde{v}\|_{L^\infty([-1/2, 0]; C^{\alpha+\beta}(B_{1/2}(0)))} \\ &\leq C_2(\|\tilde{v}\|_{L^\infty([-1, 0] \times \mathbb{R}^d)} + \|\tilde{f}\|_{L^\infty([-1, 0]; C^\beta(B_{1/2}(0)))) = C_2(\|v\|_0 + \|f_0\|_{C_b^\beta(\mathbb{R}^d)}), \end{aligned}$$

where  $C_2$  depends only on  $\|b_0\|_{C_b^\beta(\mathbb{R}^d; \mathbb{R}^d)} = \|b\|_{C_b^\beta(\mathbb{R}^d; \mathbb{R}^d)}$ ,  $\alpha$  and  $d$  and is independent of  $\lambda$ . Thus we get that  $u_\lambda \in C^{\alpha+\beta}(B_{1/2}(x_0))$  with a bound for the  $C^{\alpha+\beta}$ -norm of  $u_\lambda$  on  $B_{1/2}(x_0)$  by the quantity  $C_2(\|u_\lambda\|_0 + \|f\|_{C_b^\beta(\mathbb{R}^d)})$ . Since  $C_2$  is independent on  $x_0$  it is clear that we have  $u_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$  (cf. for instance page 434 in [24]) and the following estimate holds with  $C_3 = C_3(\|b\|_{C_b^\beta}, \alpha, d, \beta) > 0$

$$\|u_\lambda\|_{C_b^{\alpha+\beta}(\mathbb{R}^d)} \leq C_3(\|u_\lambda\|_0 + \|f\|_{C_b^\beta(\mathbb{R}^d)}).$$

By Proposition 3.2 in [24] we know that  $\lambda \|u_\lambda\|_0 \leq \|f\|_0$ . Hence we arrive at

$$\|u_\lambda\|_{C_b^{\alpha+\beta}(\mathbb{R}^d)} \leq 2C_3 \|f\|_{C_b^\beta(\mathbb{R}^d)}, \quad \lambda \geq 1. \quad (6.18)$$

*II step.* Let  $\lambda \geq 1$ . We show the existence of a  $C_b^1$ -solution to  $\lambda w - \mathcal{L}_b w = \tilde{f}$  when  $b \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$  and  $\tilde{f} \in C_b^\infty(\mathbb{R}^d)$ .

To construct the solution we use a probabilistic method (for an alternative vanishing viscosity method see Section 3.2 in [31]). Let  $(X_t^x)$  be the solution of  $dX_t = b(X_t)dt + dL_t$ ,  $X_0 = x \in \mathbb{R}^d$  and consider the associated Markov semigroup  $(R_t)$ , i.e.,  $R_t l(x) = E[l(X_t^x)]$ ,  $t \geq 0$ ,  $x \in \mathbb{R}^d$ ,  $l \in UC_b(\mathbb{R}^d)$  ( $UC_b(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$  denotes the Banach space of all uniformly continuous and bounded functions endowed with the sup-norm). Differentiating with respect to  $x$  under the expectation (using the derivative of  $X_t^x$  with respect to  $x$ , cf. [37]) it is straightforward to prove that  $R_t g \in C_b^1(\mathbb{R}^d)$ , for any  $t \geq 0$  and  $g \in C_b^1(\mathbb{R}^d)$ . For the given  $\tilde{f} \in C_b^\infty(\mathbb{R}^d)$  we define

$$\tilde{w}(x) = \tilde{w}_\lambda(x) = \int_0^\infty e^{-\lambda t} R_t \tilde{f}(x) dt, \quad x \in \mathbb{R}^d. \quad (6.19)$$

It is clear that  $\tilde{w} \in C_b(\mathbb{R}^d)$ . We now show that  $\tilde{w} \in C_b^1(\mathbb{R}^d)$  and solves our equation. To this purpose we first prove that for  $t > 0$

$$\sup_{x \in \mathbb{R}^d} |DR_t \tilde{f}(x)| \leq c(\alpha, \beta, \|Db\|_0) (t \wedge 1)^{(\beta-1)/\alpha} \|\tilde{f}\|_{C_b^\beta(\mathbb{R}^d)}. \quad (6.20)$$

Once this estimate is proved, differentiating under the integral sign in (6.19) we obtain that  $w \in C_b^1(\mathbb{R}^d)$  since  $\alpha + \beta > 1$ . Let us fix  $t \in (0, 1]$ . By Theorem 1.1 in [37] we know in particular that

$$\|DR_t g\|_0 = \sup_{x \in \mathbb{R}^d} |DR_t g(x)| \leq c(\alpha) e^{\|Db\|_0} t^{-1/\alpha} \|g\|_0, \quad g \in C_b^1(\mathbb{R}^d).$$

Using the total variation norm as in Lemma 7.1.5 of [7] we deduce that  $R_t l$  is Lipschitz continuous for any  $l \in UC_b(\mathbb{R}^d)$  and moreover  $|R_t l(x) - R_t l(y)| \leq c(\alpha) e^{\|Db\|_0} t^{-1/\alpha} |x - y| \|l\|_0$ ,  $x, y \in \mathbb{R}^d$ . By Theorem 1.1 in [37], for any  $g \in C_b^1(\mathbb{R}^d)$ , we can write the directional derivative of  $R_t g$  along  $h \in \mathbb{R}^d$  as follows:

$$D_h R_t g(x) = E[g(X_t^x) J(t, x, h)], \quad x \in \mathbb{R}^d, \quad (6.21)$$

where  $J(t, x, h)$  is a suitable random variable such that  $(E|J(t, x, h)|^2)^{1/2} \leq c(\alpha) e^{\|Db\|_0} t^{-1/\alpha} |h|$ , for any  $x \in \mathbb{R}^d$ . Let again  $l \in UC_b(\mathbb{R}^d)$ . Using mollifiers we can consider an approximating sequence  $(g_n) \subset C_b^\infty(\mathbb{R}^d)$  such that  $\|g_n - l\|_0 \rightarrow 0$  as  $n \rightarrow \infty$ . Using (6.21) when  $g$  is replaced by  $g_n$  and passing to the limit it is not difficult to prove that  $R_t l \in C_b^1(\mathbb{R}^d)$  and moreover (6.21) holds when  $g$  is replaced by  $l$  (cf. page 480 in [23]).

We have found that  $R_t : UC_b(\mathbb{R}^d) \rightarrow C_b^1(\mathbb{R}^d)$  is a linear and bounded operator and  $|DR_t l(x)| \leq c(\alpha) e^{\|Db\|_0} t^{-1/\alpha} \|l\|_0$ , for  $x \in \mathbb{R}^d$ ,  $l \in UC_b(\mathbb{R}^d)$ . Moreover,  $R_t : C_b^1(\mathbb{R}^d) \rightarrow C_b^1(\mathbb{R}^d)$  is linear and bounded and  $|DR_t g(x)| \leq e^{\|Db\|_0} \|Dg\|_0$ , for  $x \in \mathbb{R}^d$ ,  $g \in C_b^1(\mathbb{R}^d)$ . To prove such estimate we fix  $h \in \mathbb{R}^d$  and differentiate  $R_t g(x)$  with respect to  $x$  along the direction  $h$ . One can show that

$$D_h E[g(X_t^x)] = E[Dg(X_t^x) \eta_t] \quad (6.22)$$

where  $\eta_t = D_h X_t^x$  solves  $\eta_t = h + \int_0^t Db(X_s^x) \eta_s ds$ ,  $t \geq 0$ ,  $P$ -a.s.. Note that  $|D_h X_t^x| \leq |h| e^{\|Db\|_0 t}$  by the Gronwall lemma (cf. page 1211 in [37]).

By interpolation techniques we know that  $(UC_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d))_{\beta, \infty} = C_b^\beta(\mathbb{R}^d)$ , for  $\beta \in (0, 1)$  (cf. [22, Chapter 1] and the proof of Theorem 3.3 in [24]); it follows that

for any  $t \in (0, 1]$  we have that  $R_t : C_b^\beta(\mathbb{R}^d) \rightarrow C_b^1(\mathbb{R}^d)$  is linear and bounded and  $|DR_t f(x)| \leq c(\alpha, \beta) e^{\|Db\|_0} t^{(\beta-1)/\alpha} \|f\|_{C_b^\beta}$ , for any  $x \in \mathbb{R}^d$ ,  $f \in C_b^\beta(\mathbb{R}^d)$ .

We have verified (6.20) when  $t \in (0, 1]$ . If  $t > 1$  we use a standard argument based on the semigroup property and get, for any  $x \in \mathbb{R}^d$ ,  $|DR_t \tilde{f}(x)| = |DR_1(R_{t-1} \tilde{f})(x)| \leq c(\alpha) e^{\|Db\|_0} \|R_{t-1} \tilde{f}\|_0 \leq c(\alpha) e^{\|Db\|_0} \|\tilde{f}\|_0$ . Thus (6.20) holds and we know that  $\tilde{w} \in C_b^1(\mathbb{R}^d)$ . To prove that  $\tilde{w}$  is a solution we first establish the identity

$$\partial_t(R_t \tilde{f})(s, x) = R_s(\mathcal{L}_b \tilde{f})(x) = \mathcal{L}_b(R_s \tilde{f})(x), \quad s \geq 0, x \in \mathbb{R}^d. \quad (6.23)$$

By using Ito's formula (see [20, Section 2.3]) and taking the expectation we find  $E[\tilde{f}(X_{s+h}^x)] - E[\tilde{f}(X_s^x)] = \int_s^{s+h} E[(\mathcal{L}_b \tilde{f})(X_r^x)] dr$ , for  $h \in \mathbb{R}$  such that  $s+h > 0$ . It follows that, for  $x \in \mathbb{R}^d$ ,

$$\partial_t(R_t \tilde{f})(s, x) = \lim_{h \rightarrow 0} h^{-1} (R_{s+h} \tilde{f}(x) - R_s \tilde{f}(x)) = R_s(\mathcal{L}_b \tilde{f})(x), \quad s > 0, \quad (6.24)$$

$$\text{and } \lim_{h \rightarrow 0^+} h^{-1} (R_h \tilde{f}(x) - \tilde{f}(x)) = \mathcal{L}_b \tilde{f}(x). \quad (6.25)$$

If  $s > 0$  by (6.25) we get  $\lim_{h \rightarrow 0^+} \frac{R_h(R_s \tilde{f})(x) - R_s \tilde{f}(x)}{h} = \mathcal{L}_b(R_s \tilde{f})(x)$  when  $\tilde{f}$  in (6.25) is replaced by  $R_s \tilde{f}$ . By the semigroup law, the last limit and (6.24) coincide and so (6.23) holds. To check that  $\tilde{w}$  verifies  $\lambda \tilde{w} - \mathcal{L}_b \tilde{w} = \tilde{f}$  we use (6.20) and (6.23). First by the Fubini theorem we have

$$\mathcal{L}_b \tilde{w}(x) = \int_0^\infty e^{-\lambda t} \mathcal{L}_b(R_t \tilde{f})(x) dt = \int_0^\infty e^{-\lambda t} R_t(\mathcal{L}_b \tilde{f})(x) dt.$$

By (6.23) it follows that, for any  $x \in \mathbb{R}^d$ ,  $\mathcal{L}_b \tilde{w}(x) = \int_0^\infty e^{-\lambda t} \frac{d}{dt}(R_t \tilde{f}(x)) dt$ . Integrating by parts, we get the assertion.

*III step.* Let  $\lambda \geq 1$ . We prove the existence of a  $C_b^{\alpha+\beta}$ -solution to  $\lambda w - \mathcal{L}_b w = f$  on  $\mathbb{R}^d$  when  $b \in C_b^\beta(\mathbb{R}^d; \mathbb{R}^d)$  and  $f \in C_b^\beta(\mathbb{R}^d)$ ,  $\alpha + \beta > 1$ , and show (6.17).

Using convolution with mollifiers and possibly passing to subsequences (see, for instance, page 431 in [24]) one can consider operators  $\mathcal{L}_{b_n}$  with drifts  $b_n \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$  such that  $\|b_n\|_{C_b^\beta} \leq \|b\|_{C_b^\beta}$ ,  $n \geq 1$ , and  $b_n \rightarrow b$  in  $C^{\beta'}(K; \mathbb{R}^d)$  for any compact set  $K \subset \mathbb{R}^d$  and  $\beta' \in (0, \beta)$ . Similarly one can construct  $(f_n) \subset C_b^\infty(\mathbb{R}^d)$  such that  $\|f_n\|_{C_b^\beta} \leq \|f\|_{C_b^\beta}$ ,  $n \geq 1$ , and  $f_n \rightarrow f$  in  $C^{\beta'}(K)$  for any compact set  $K \subset \mathbb{R}^d$  and  $\beta' \in (0, \beta)$ . By II step there exist  $C_b^1$ -solutions  $w_n$  to  $\mathcal{L}_{b_n} w_n = \lambda w_n - f_n$ ,  $n \geq 1$ . By step I we know that  $w_n \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ ,  $n \geq 1$ , with the estimate

$$\|w_n\|_{C_b^{\alpha+\beta}(\mathbb{R}^d)} \leq 2C_3 \|f\|_{C_b^\beta(\mathbb{R}^d)}, \quad (6.26)$$

( $C_3 = C_3(\|b\|_{C_b^\beta}, \alpha, \beta, d)$  is independent of  $\lambda$  and  $n$ ). Possibly passing to a subsequence still denoted with  $(w_n)$ , we have that  $w_n \rightarrow w$  in  $C^{\alpha+\beta'}(K)$ , for any compact set  $K \subset \mathbb{R}^d$  with  $\beta' > 0$  such that  $1 < \alpha + \beta' < \alpha + \beta$ . Moreover, (6.26) holds with  $w_n$  replaced by  $w$ . We can easily pass to the limit in each term of  $\lambda w_n(x) - \mathcal{L} w_n(x) - b_n(x) \cdot Dw_n(x) = f_n(x)$  as  $n \rightarrow \infty$  and obtain that  $w$  solves our equation.

*IV step.* We prove the final assertion.

We already know that there exists a unique solution  $w_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$  and that (6.17) holds. To complete the proof we argue as in the final part of the proof of Theorem 6.7. By the interpolatory estimate  $\|Dw_\lambda\|_0 \leq N(\alpha, \beta, d)[Dw_\lambda]_{C_b^{\alpha+\beta-1}}^{\frac{1}{\alpha+\beta}} \|w_\lambda\|_0^{\frac{\alpha+\beta-1}{\alpha+\beta}}$ , we obtain easily that  $\|Dw_\lambda\|_0 < 1/3$  for  $\lambda \geq \lambda_0(d, \|b\|_{C_b^\beta}, \alpha, \beta)$ . ■

*Proof of Theorem 6.10.* As in the proof of Theorem 6.8 we verify the assumptions of Theorem 5.1. Note that Hypothesis 2 holds since  $\int_{\{|x|>1\}} \frac{|y|^\theta}{|y|^{d+\alpha}} dy < \infty$ , for any  $\theta \in (0, \alpha)$ . In order to check Hypothesis 1 we argue as in the proof of Theorem 6.8 (using Theorems 6.11 and 6.6; recall that  $\alpha = \alpha_0$ ). The proof is complete. ■

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